Structural analysis of approximate pattern matching and algorithmic applications

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Roadmap

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Approximate pattern matching under Hamming and edit distance

- Structural results: either the pattern is almost periodic or it only has a few approximate occurrences.
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Implementations of PILLAR operations in different settings: the standard setting and the compressed setting.

Further improvements to the obtained algorithm for approximate pattern matching under edit distance.

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We say that a string S is periodic if $\mathrm{per}(S) \leq |S|/2$.

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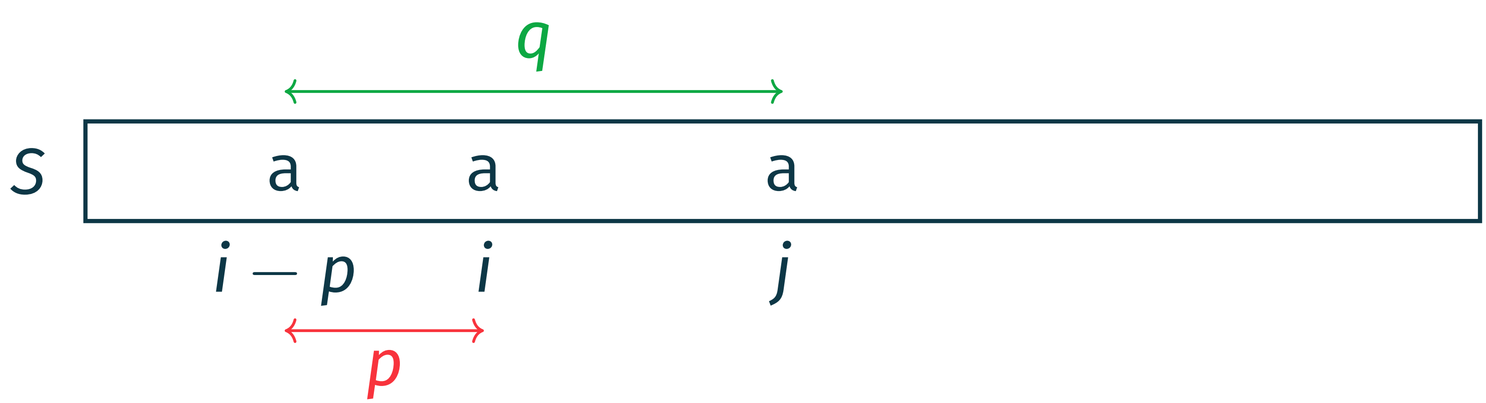

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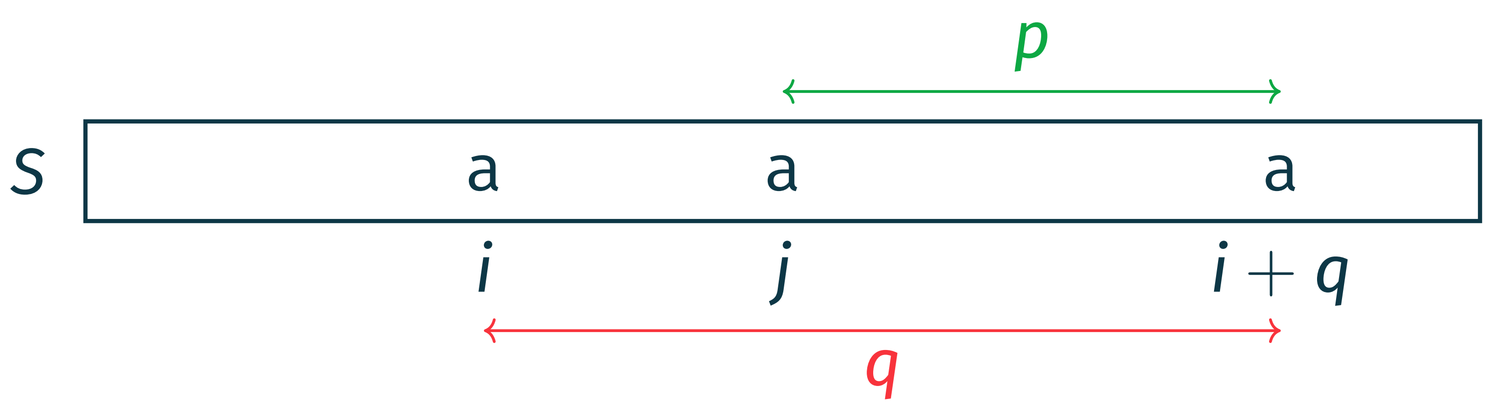

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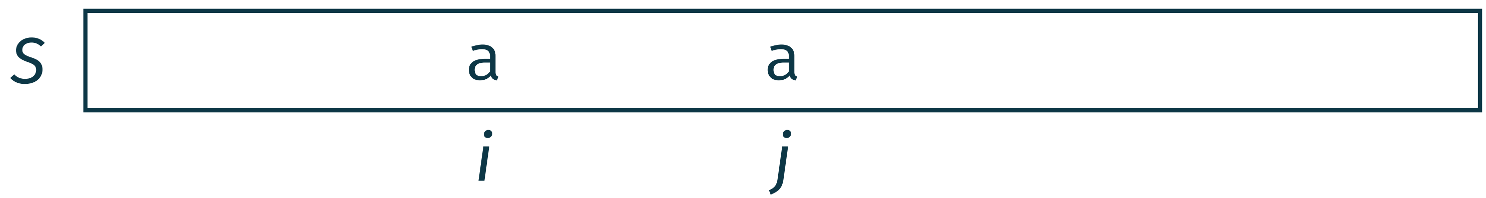

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$(p, q) \rightarrow (p, q - p) \rightarrow \cdots \rightarrow (p, q \bmod p)$

This yields $gcd(p, q)$ as in Euclid’s algorithm (for computing the gcd of p and q).

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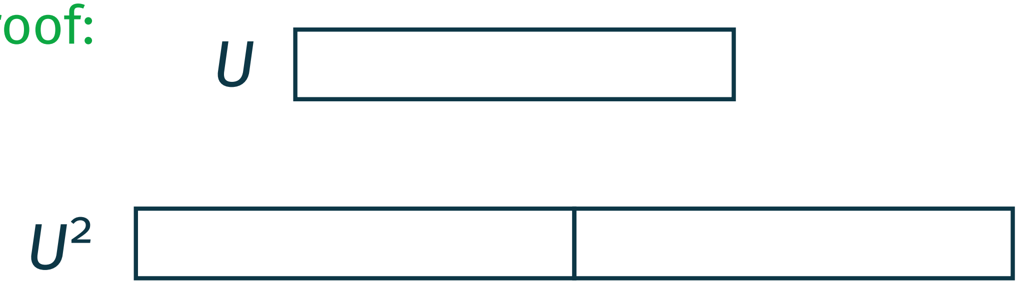

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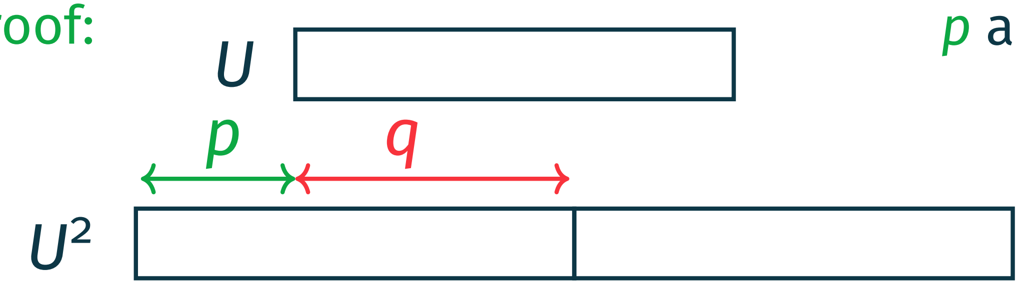

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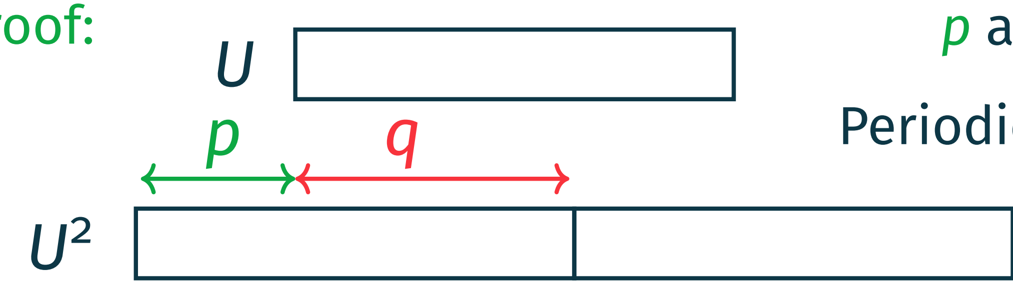

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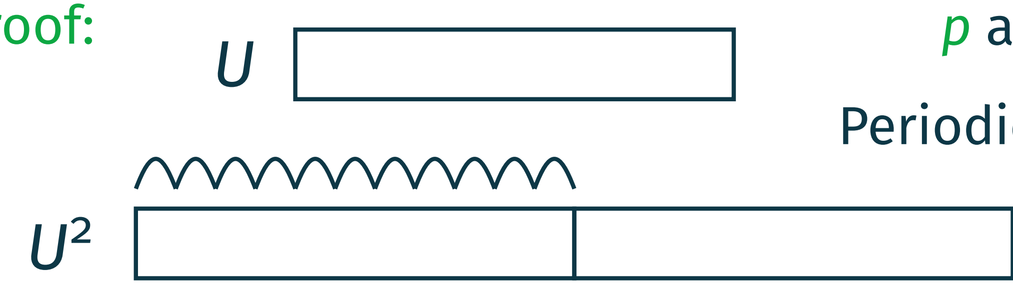

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The solutions to many string algorithmic problems distinguish between the aperiodic and periodic cases.

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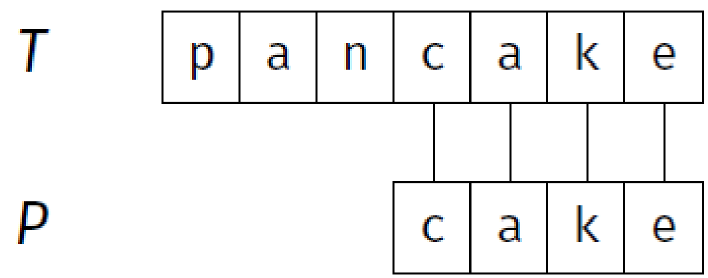

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What if there is no periodicity whatsoever in the pattern?

(That is, what if none of P's substrings have a non-trivial period?)

For example, what if $P = \mathtt{a\,b\,c\,d\,e\,f\,g\,h\,i\,j\,k}$?

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Each successful letter comparison consumes a letter of T, while each unsuccessful letter comparison shifts P. $\rightarrow \mathcal{O}(n)$ time!

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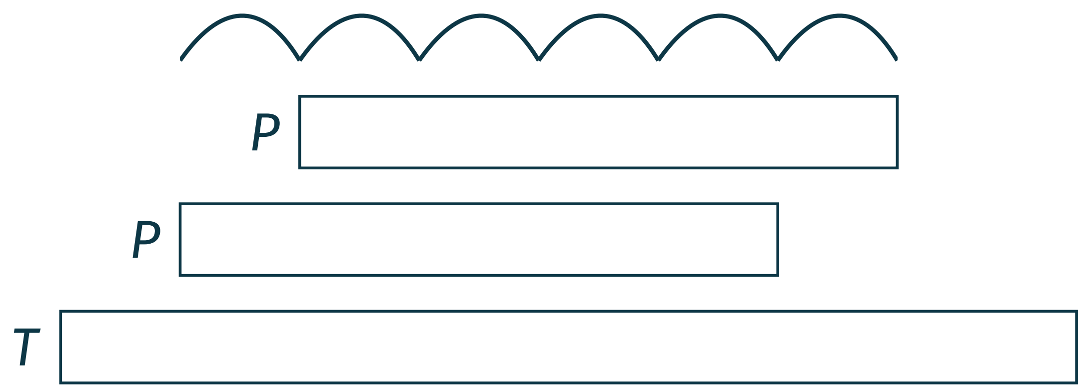

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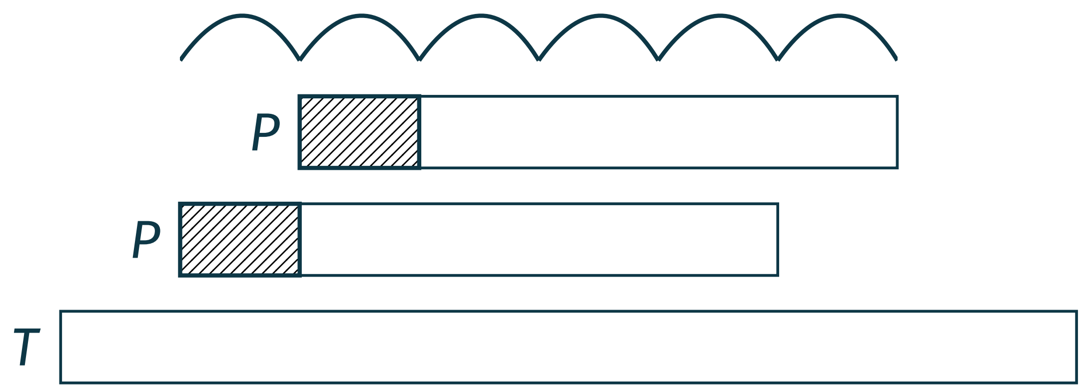

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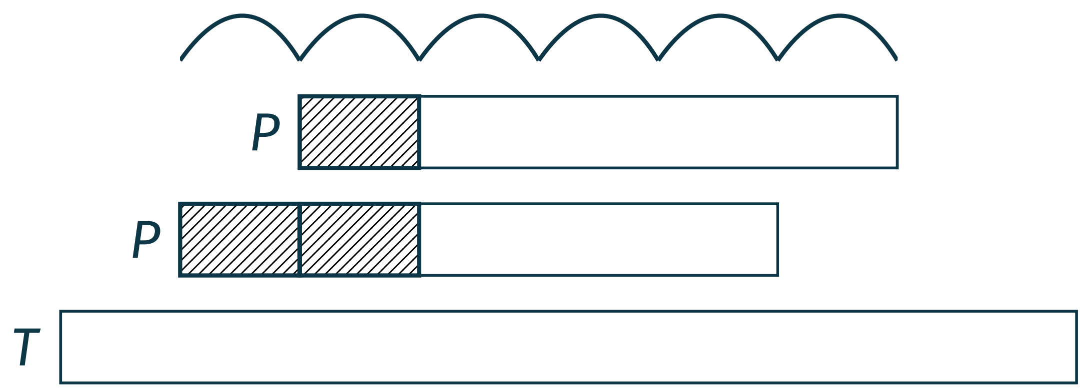

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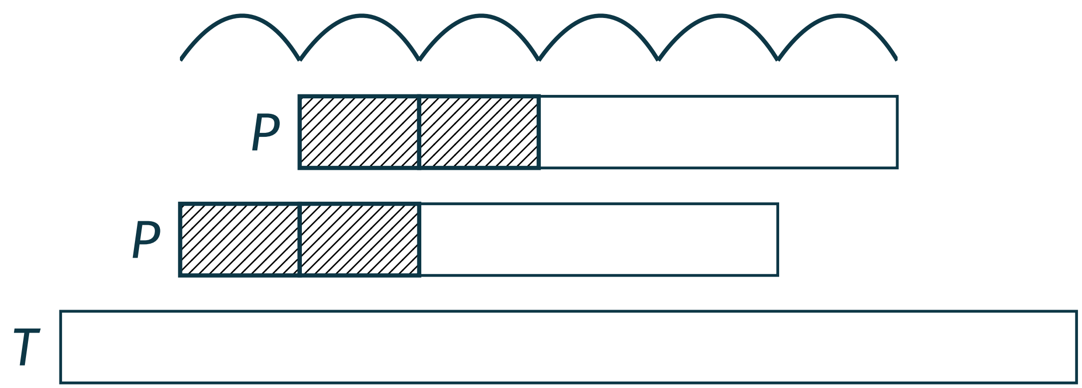

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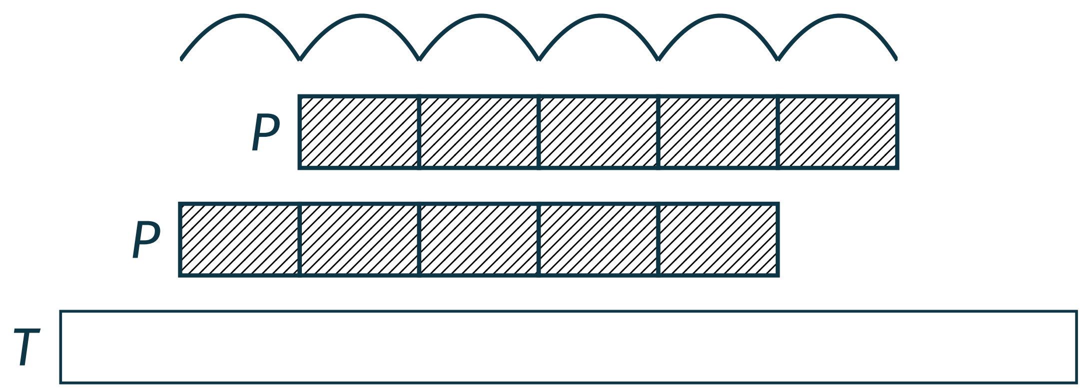

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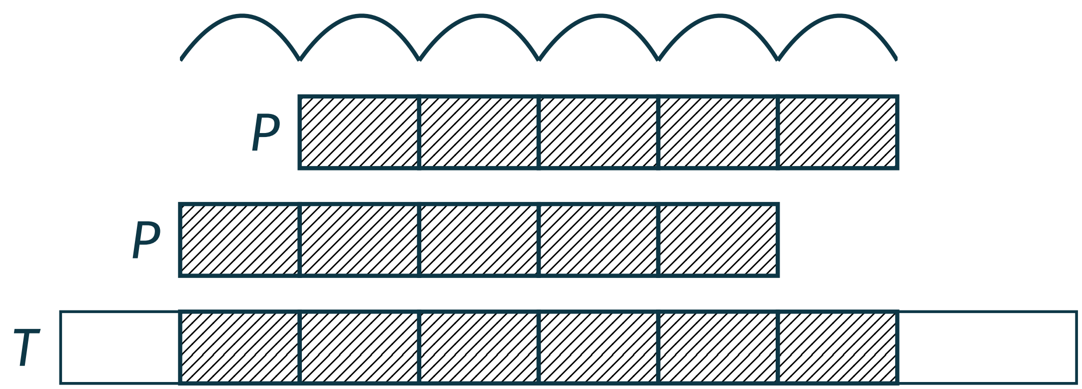

The fragment of T spanned by P's occurrences is periodic as well.

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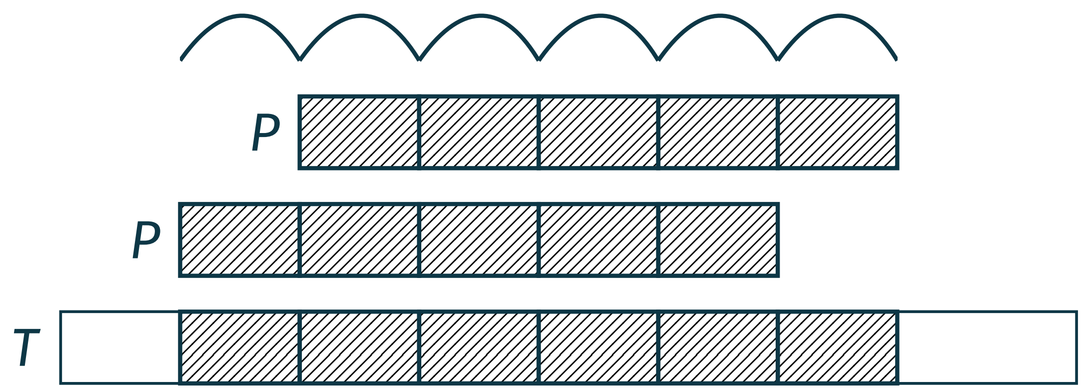

The standard trick: Our assumption on the length of the text is not restrictive. If the text is much longer that the pattern, we can always consider separately $\mathcal{O}(n/m)$ fragments of T of length $\leq {}^3\!/_2\, m$ that overlap by $m - 1$ positions.

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We will discuss the structure of approximate pattern matching under each of these metrics and see how the structural analysis yields efficient algorithms in several settings.

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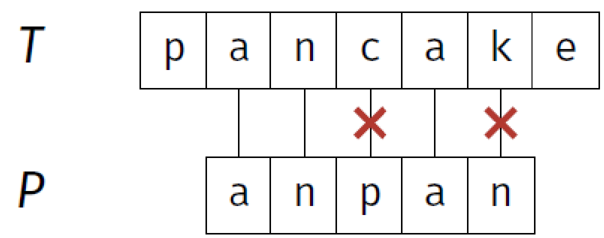

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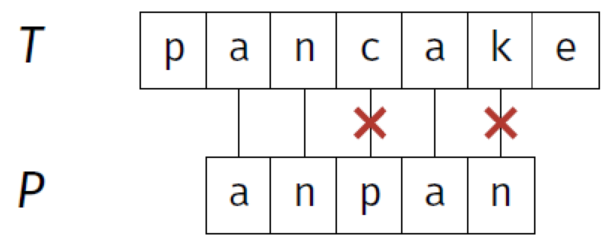

Pattern Matching under Edit Distance

Given a text T, a pattern P, and an integer threshold k, compute the (starting positions of) substrings of T that are at edit distance at most k from P.

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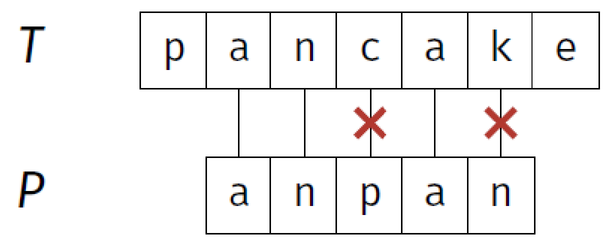
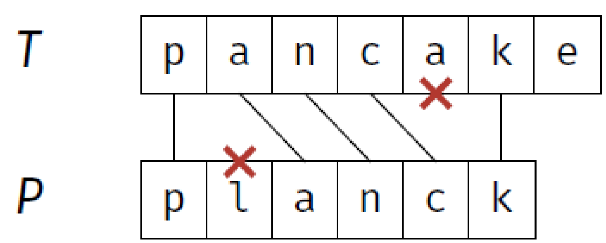


Pattern Matching under Edit Distance

Given a text T, a pattern P, and an integer threshold k, compute the (starting positions of) substrings of T that are at edit distance at most k from P.



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The first explicitly stated structural result (for Hamming distance) was proved by Bringmann, Künnemann, and Wellnitz in 2019.

This result was tightened and extended to also cover approximate pattern matching under the edit distance by C., Kociumaka, and Wellnitz in 2020.

Partitioning the pattern and extending seeds

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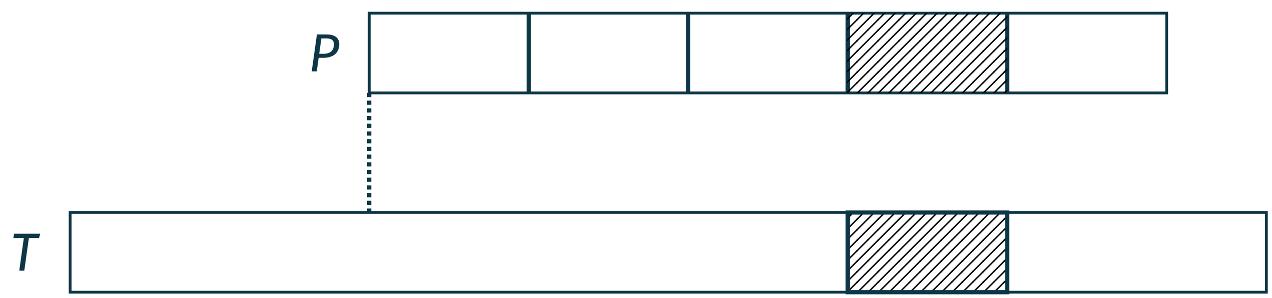
Observation: Let us partition P into $k+1$ (roughly) equal chunks, each of length $\approx m/(k+1)$. In any approximate match of P in T, at least one of the chunks must be matched exactly.



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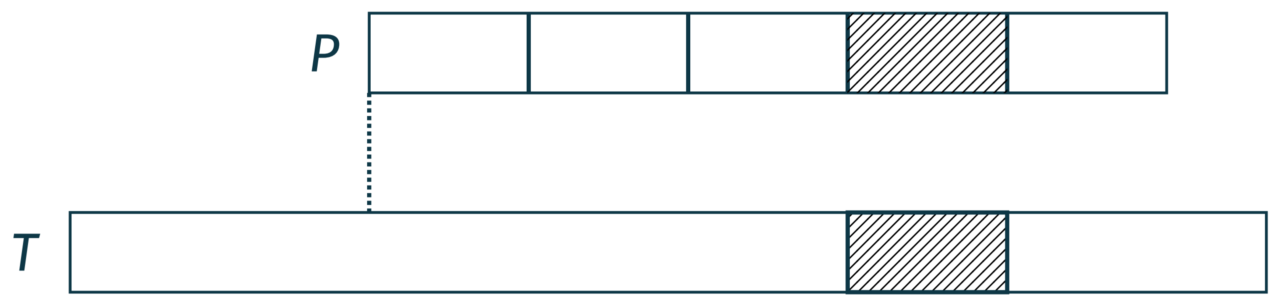

Algorithm strategy: Find the exact matches of each chunk in T (called seeds) and try to extend them to approximate matches of P.

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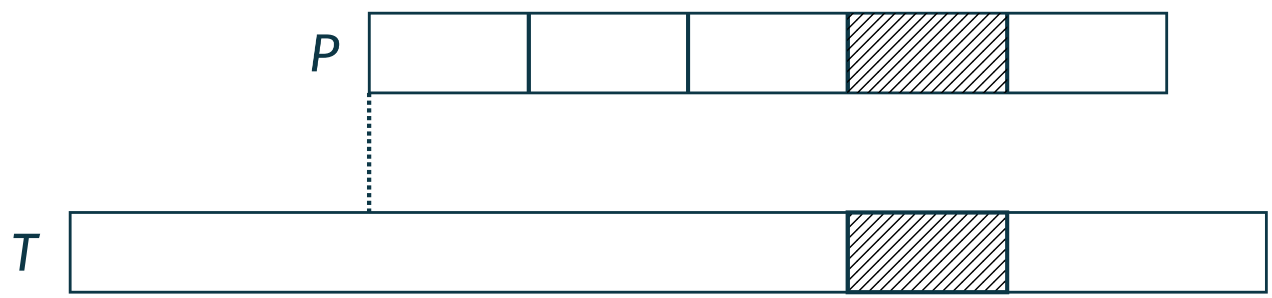

Algorithm strategy: Find the exact matches of each chunk in T (called seeds) and try to extend them to approximate matches of P.

Observation: If a chunk P_i is aperiodic, its occurrences cannot overlap by more than $|P_i|/2$ positions $\Rightarrow$ at most $n/(|P_i|/2)$ occurrences, which is $\mathcal{O}(k \cdot n/m)$.

The Marking Trick (for Hamming distance and $n \leq 2m$)

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Complexity of the seeding technique in an aperiodic case:

- $\mathcal{O}(k)$ calls to exact pattern matching, one for each chunk;
- $\mathcal{O}(k^2)$ attempts to extend a seed ($\mathcal{O}(k)$ for each of the $\mathcal{O}(k)$ chunks).

  Each seed gives a candidate starting position for an approximate occurrence.

  This gives us an $\mathcal{O}(k^2)$ bound on approximate occurrences in the case where all chunks are aperiodic!

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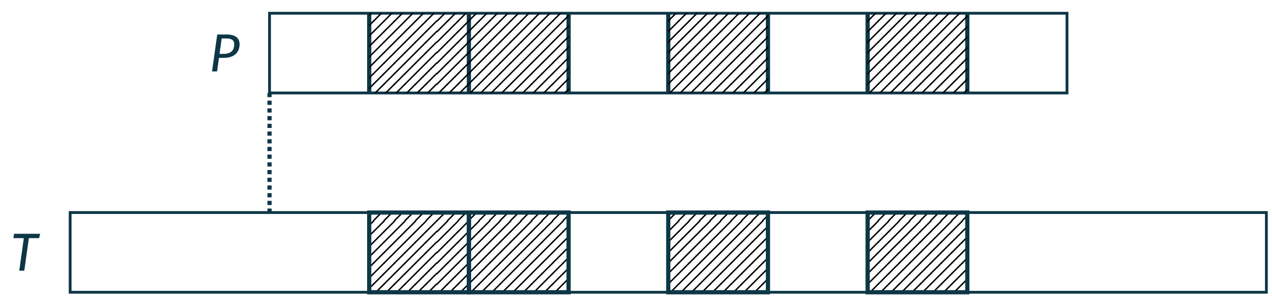
Marking trick: Partition P into $2k$ chunks, each of length $\approx m/(2k)$. In any approximate match of P in T, at least k of the chunks must be matched exactly.



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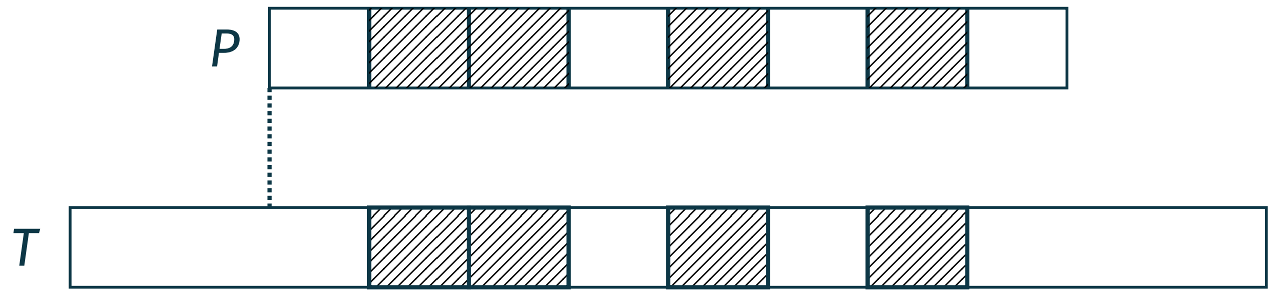

For each exact match of a chunk, give a mark to the candidate starting position of an approximate occurrence in T. Then, we only need to verify candidate starting positions with $\geq k$ marks. These are $\mathcal{O}(k)$ as we have $\mathcal{O}(k^2)$ marks overall.

The First Structural Result for Hamming Distance [BKW'19]

Theorem (Bringmann-Künnemann-Wellnitz, SODA 2019)

Given a pattern P of length m, a text T of length $n \leq 2m$, and a threshold $k \leq m$, at least one of the following holds:

- The number of k-mismatch occurrences of P in T is $O(k^2)$.
- The pattern P is almost periodic (at HD $\leq 6k$ to a string with period $O(m/k)$).

Proof Overview

Structural Theorem (HD) [Bringmann-Künnemann-Wellnitz, SODA 2019]

For a pattern P of length m and a text T of length $\le 2|P|$, at least one of the following holds:

- The number of k-mismatch occurrences of P in T is at most $1000k^2$.
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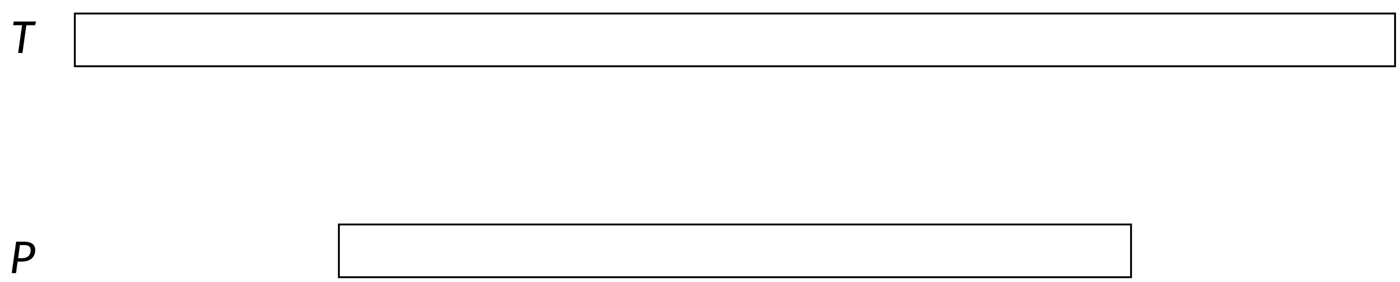

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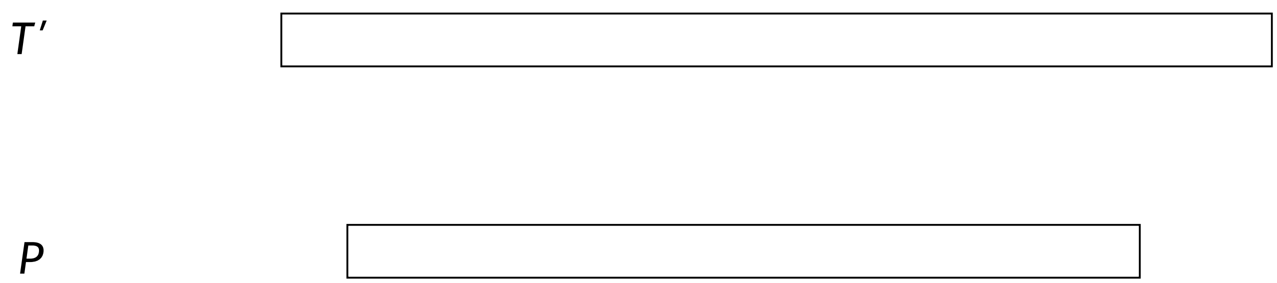

- Consider T': shortest substring of T that contains all k-mismatch occurrences of P.

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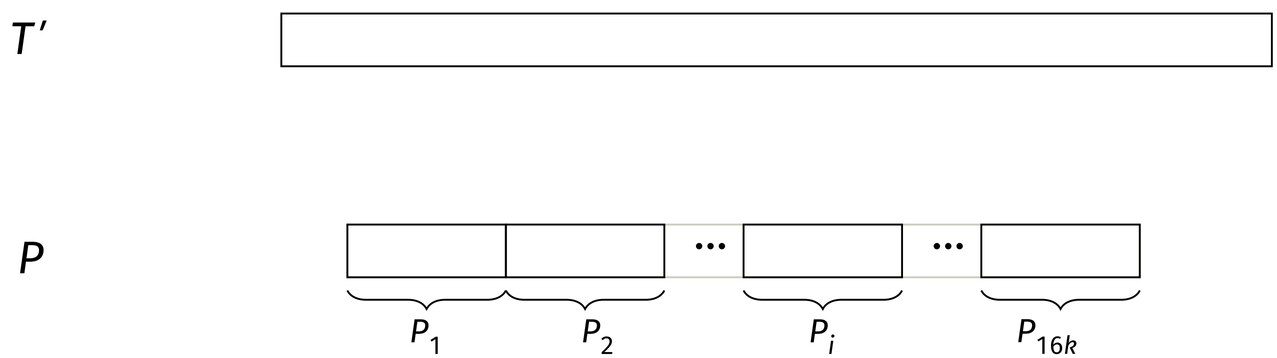

- Partition P into $16k$ parts P_i of (roughly) equal length.

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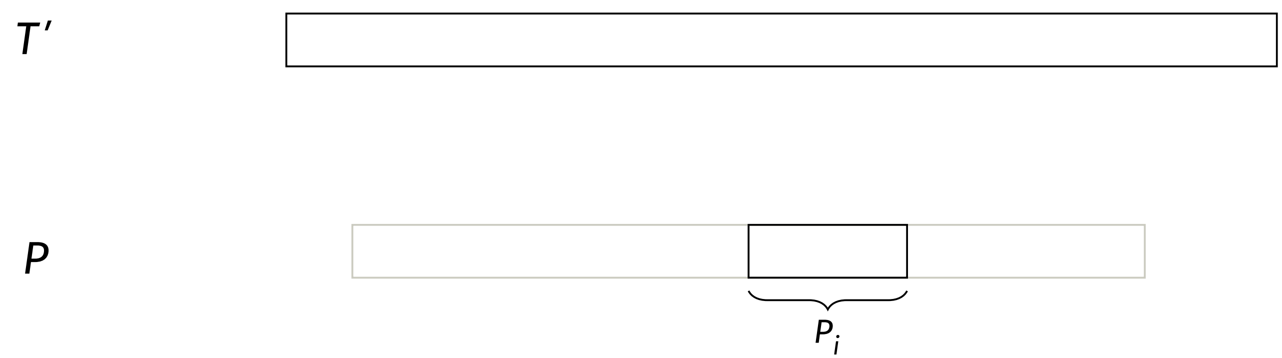

- In any k-mismatch occurrence of P in T' at least one of the P_i's must be matched **exactly**. Fix some P_i and assume that it is periodic; otherwise it only has $O(k)$ occurrences in T.

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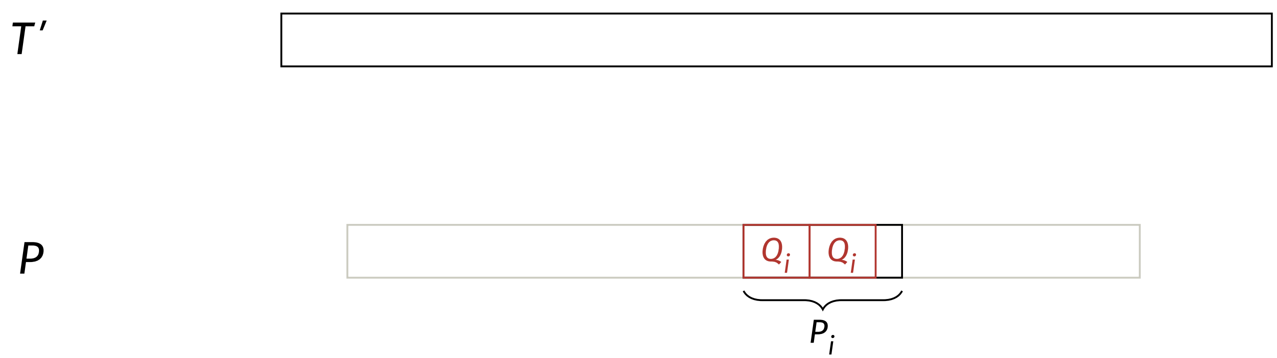

- Let Q_i be the prefix of P_i whose length is equal to the period of P_i.

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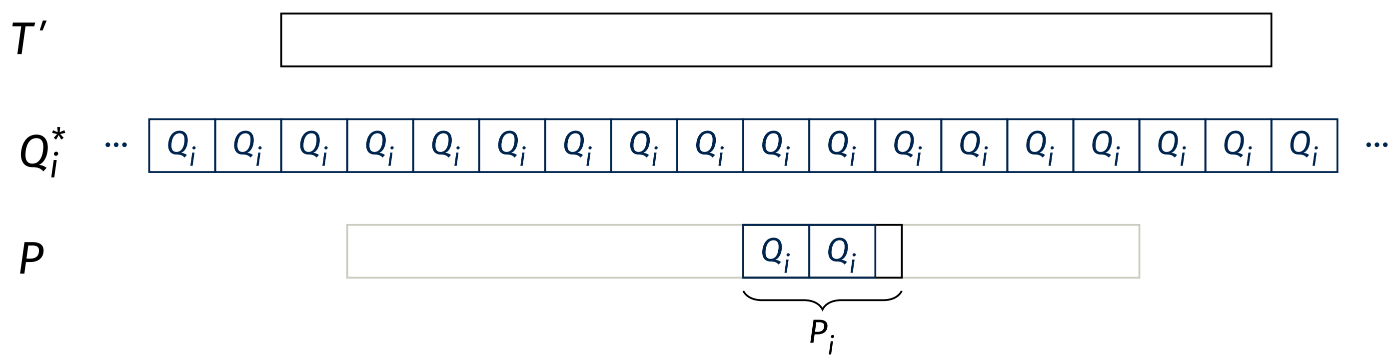

- Find the first $3k$ mismatches between P and Q_i^* before and after P_i.
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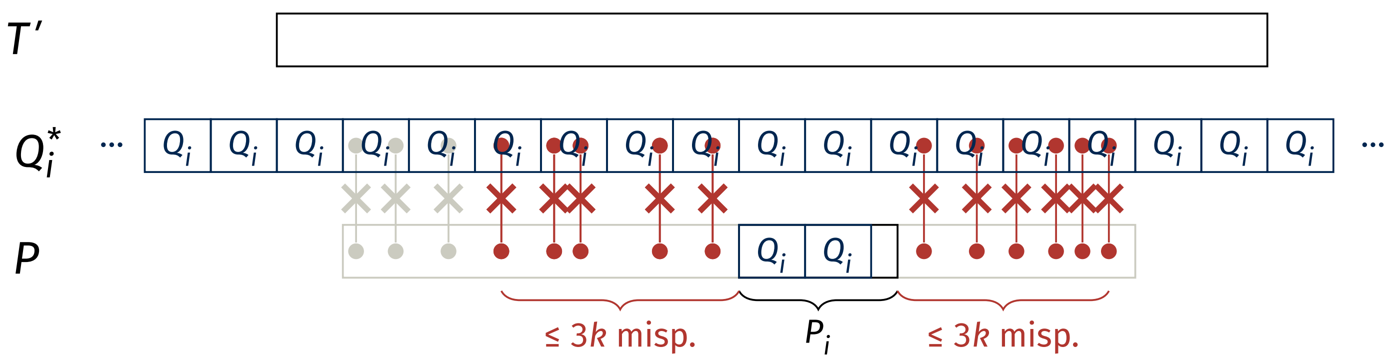

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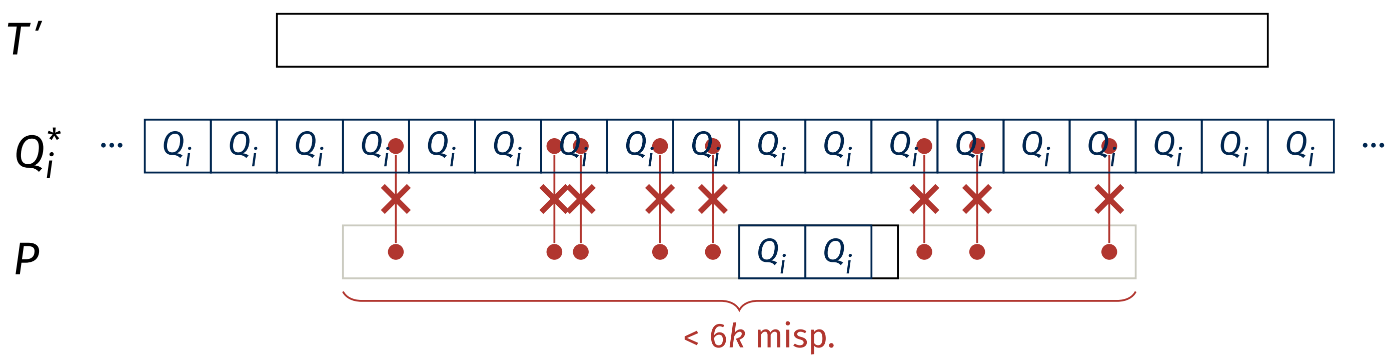

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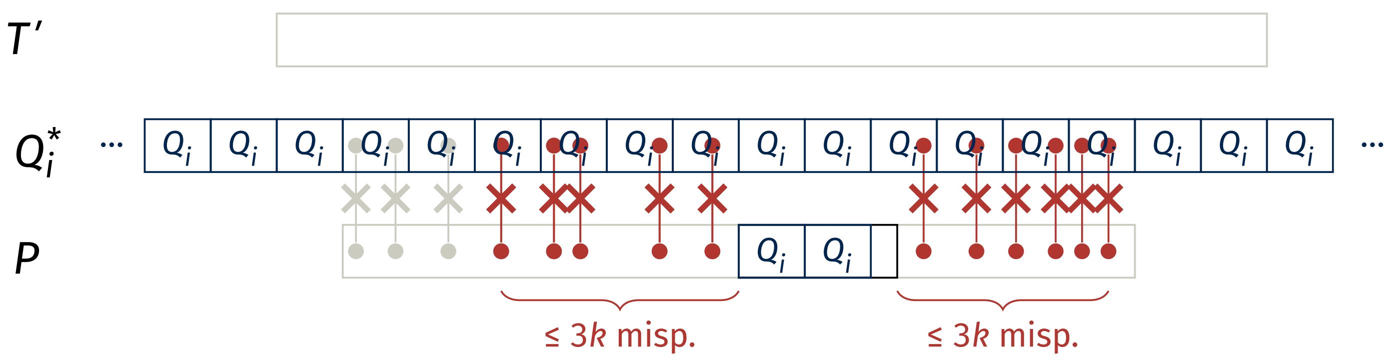

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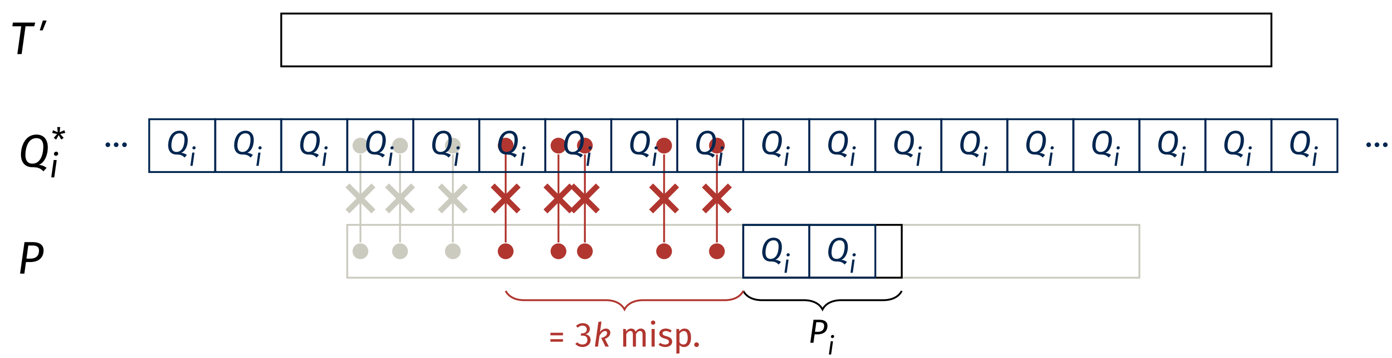

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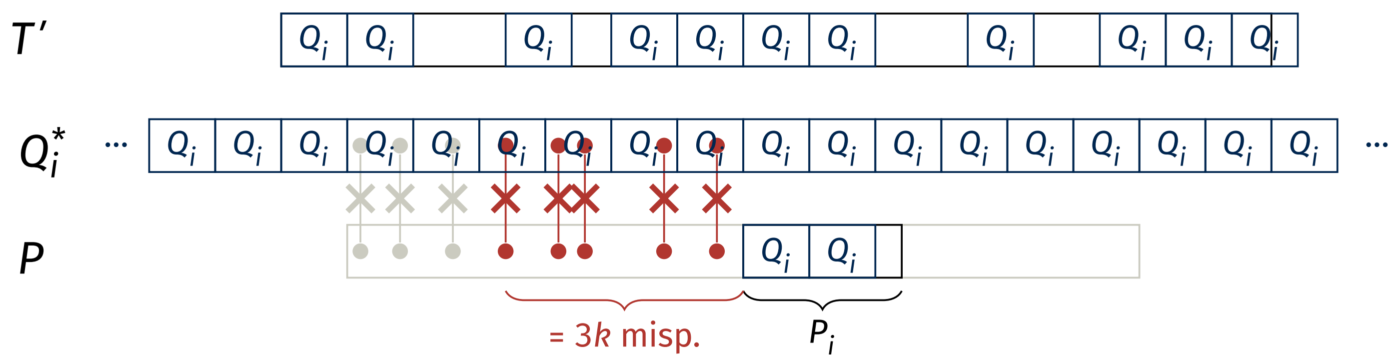

- Consider the occurrences of Q_i in T'.

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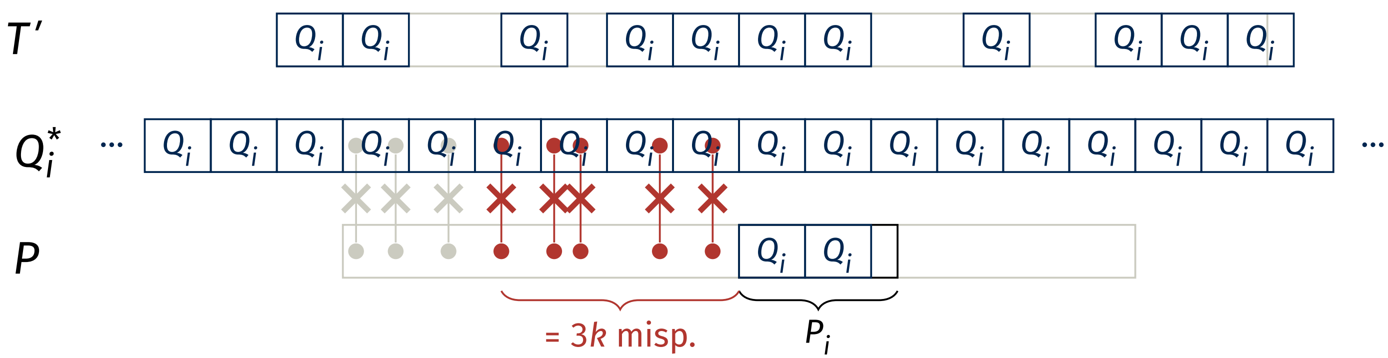

Problem

Up to $O(m)$ exact matches of Q_i in T'.

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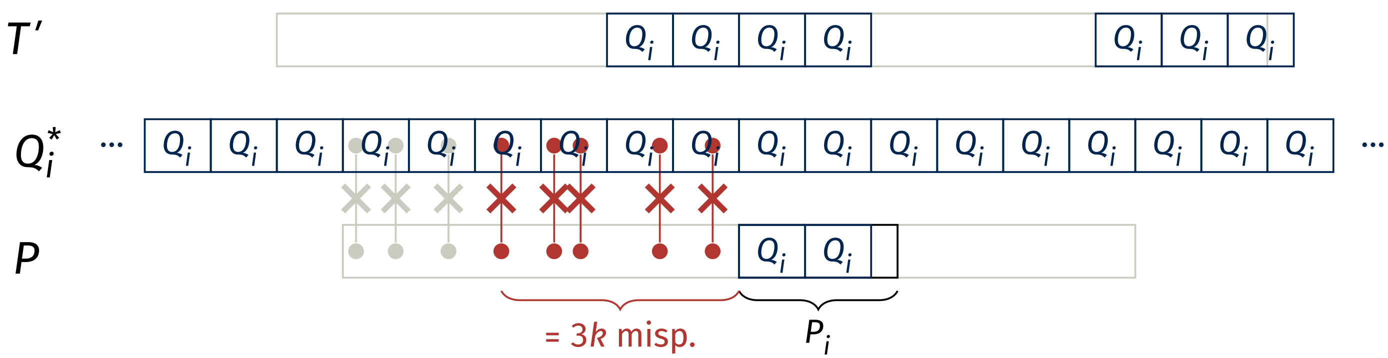

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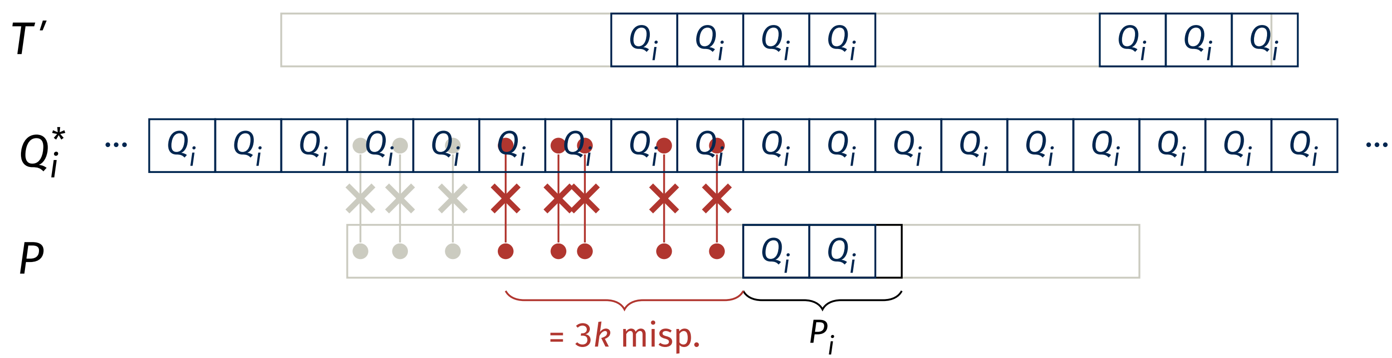

- Consider **power stretches** of Q_i in T' of length $\geq |P_i|$
  $\rightsquigarrow$ at most $150k$ different power stretches.

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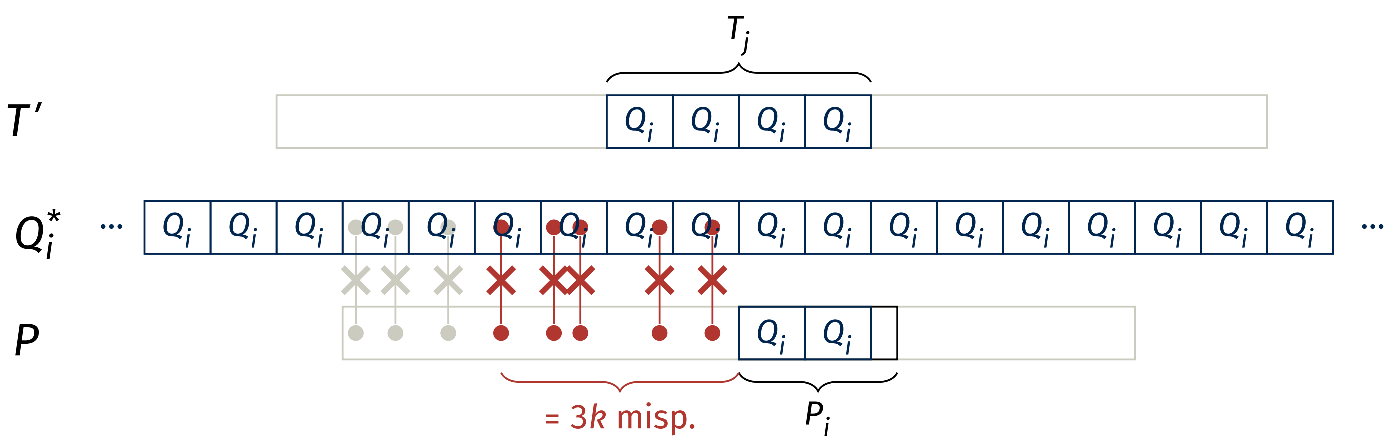

- Fix a power stretch T_j of Q_i in T'.

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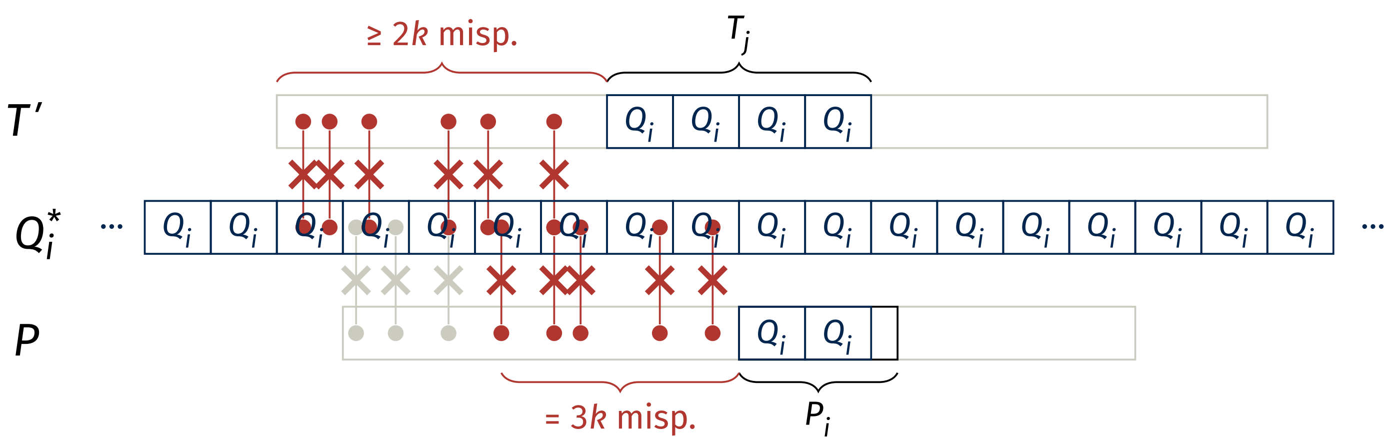

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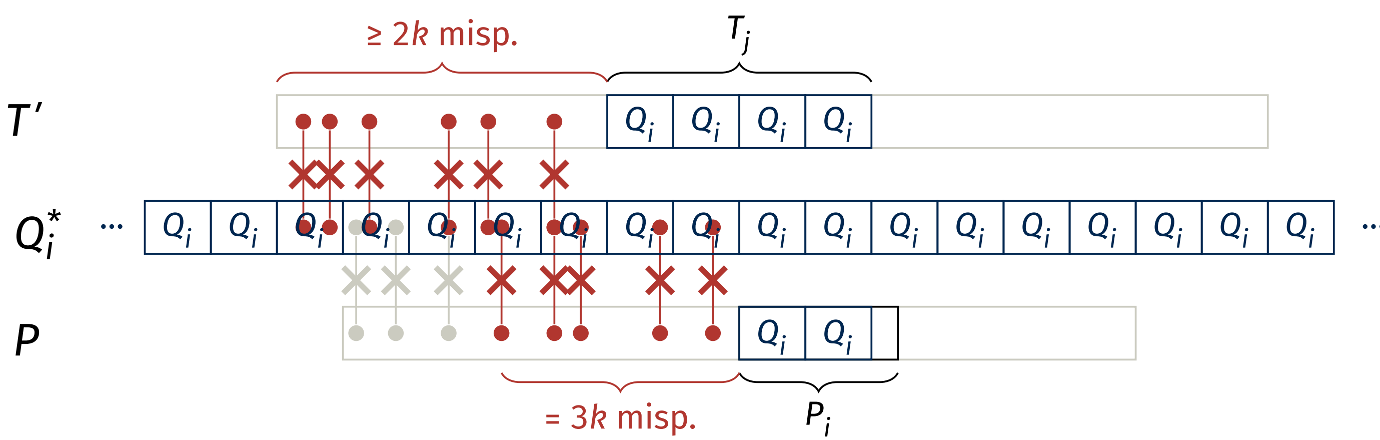

Insight

Must align at least one misperiod. (For intuition, consider the case where $T' = Q_i^t$ for some integer t. Then, for any exact match of P_i all misperiods in P yield mismatches between P and T'!)

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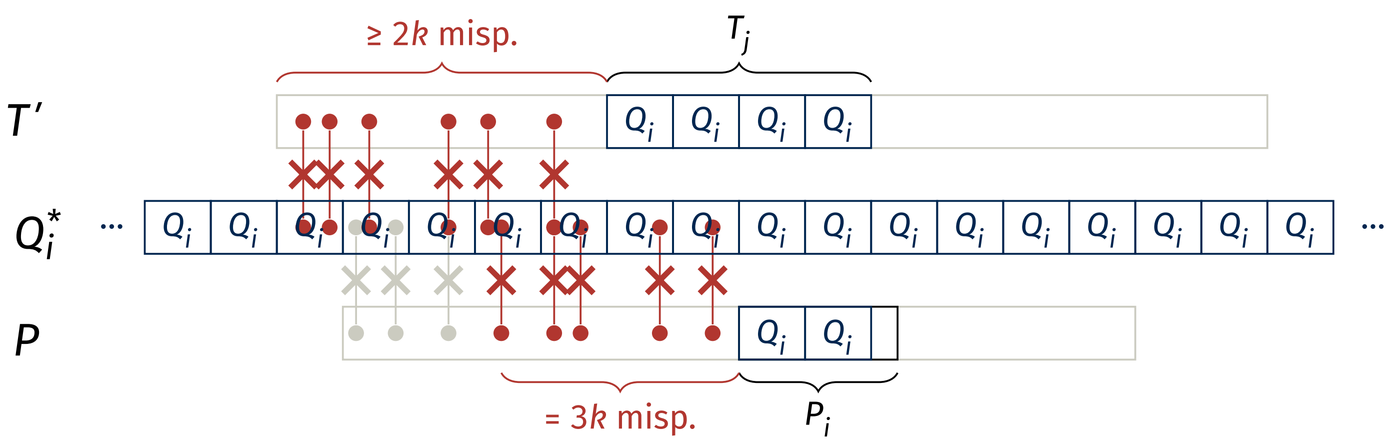

Almost there

At most $O(k^4)$ k-mismatch occurrences: $O(k)$ choices for P_i, $O(k)$ choices for a power stretch, $O(k^2)$ pairs of aligned misperiods per combination.

Summary of the proof and a challenge: $O(k^2)$ bound using the marking trick?

- Partition the pattern into $16k$ chunks P_i of length $\approx m/16k$.
- Each aperiodic chunk has $O(k)$ exact occurrences. Each such occurrence gives a single candidate starting position for a k-mismatch occurrence of P. $O(k^2)$ overall.
- For each periodic chunk P_i, extend the periodicity in both sides, allowing $3k$ misp.
  - If all of P gets covered, we conclude that the pattern is almost periodic.
  - Else, for each of the $O(k)$ power stretches with period $\mathrm{per}(P_i)$ that contain occurrences of P_i, extend the periodicity in both sides, allowing $2k$ misperiods.
    Crucial observation: in any k-error occurrence that matches P_i exactly, a misperiod in P must be aligned with a misperiod in T.

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Nearly there

We have $O(k^3)$ k-mismatch occurrences: $O(k)$ choices for P_i, $O(k)$ choices for a power stretch, $O(k^2)$ pairs of aligned misp. per combination. Need $\geq k$ pairs of alinged misp.

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    At least k P_j's must nominate any potential starting position.

Finally there, maybe

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Is this result tight?

Can we find a pattern P of length m that is not almost periodic and has $O(k^2)$ k-mismatch occurrences in a text T of length $n \leq 2m$?

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Construct an example where the pattern P

- is not at Hamming distance $O(k)$ from any string with period $O(m/k)$,
- has $\Omega(k)$ k-mismatch occurrences in T.

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Exercise

Construct an example where the pattern P

- is not periodic,
- it is at Hamming distance $O(k)$ from a string with period $O(m/k)$,
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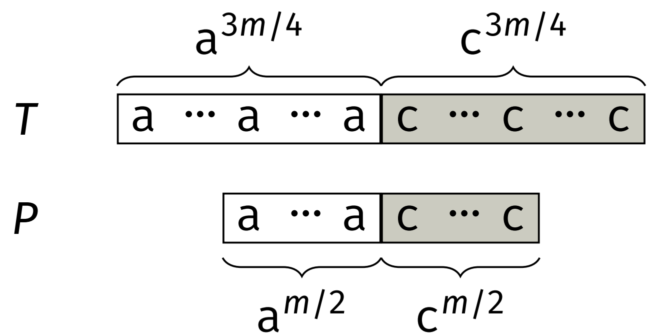
Examples



- Both P and T far from periodic, but there are $2k + 1$ k-mismatch occurrences of P in T.

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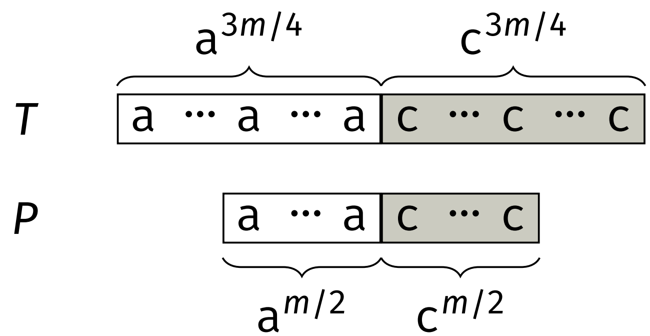

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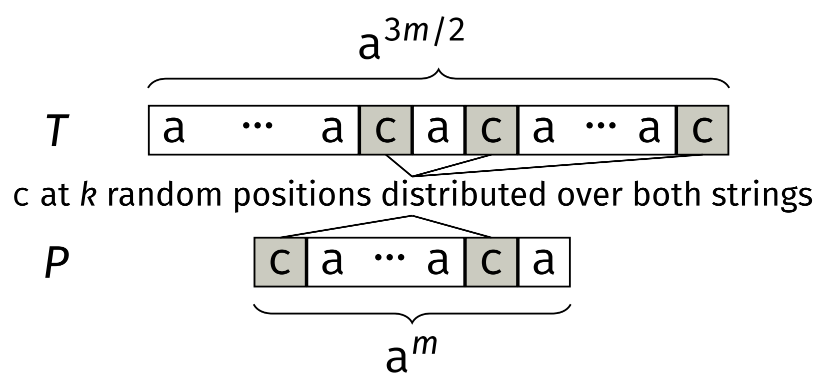

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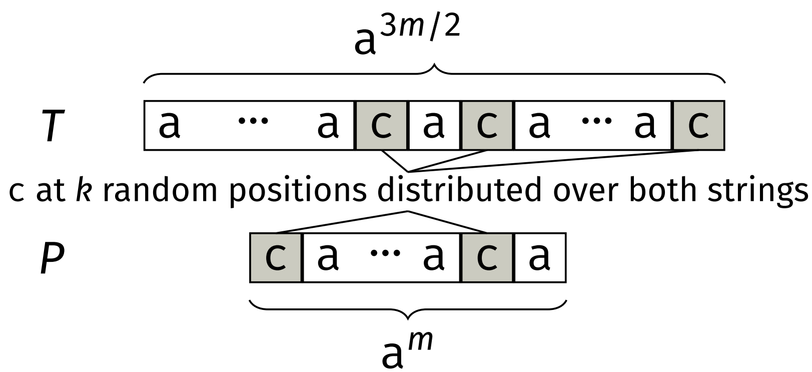

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In the approach we just saw: The pattern P is independently aligned with a substring of Q_i^∞ for each chunk P_i.

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Idea: Analyse the (periodic structure of the) pattern as a whole.

Improved Structural Results for PM with Mismatches

Structural Theorem (HD) [C.-Kociumaka-Wellnitz, FOCS 2020]

Given a pattern P of length m, a text T of length $n \leq {}^3/_2\, m$, and a threshold $k \leq m$, at least one of the following holds:

- The number of k-mismatch occurrences of P in T is $\cancel{O(k^2)}$ $O(k)$.
- The pattern P is almost periodic (at HD $\cancel{\leq 6k}$ $< 2k$ to a string Q with period $O(m/k)$).

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For each string P of length m, at least one of the following holds:

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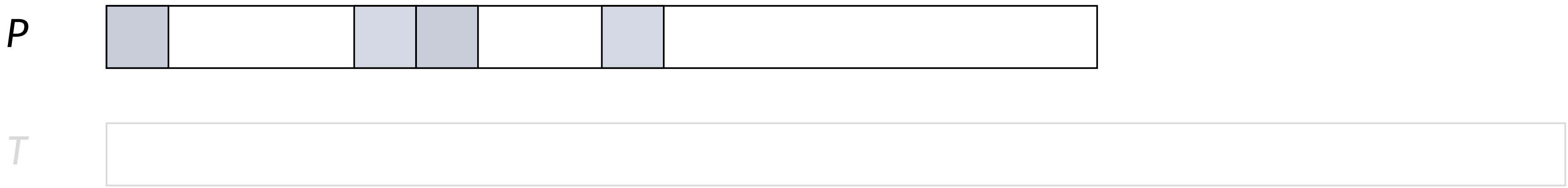
Observation [BKW'19, refined]

If P contains $\ge 2k$ disjoint breaks, there are $O(k)$ k-mismatch occ's of P in T.

Consider an example with $k = 2$.



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Let us focus on a single break B.

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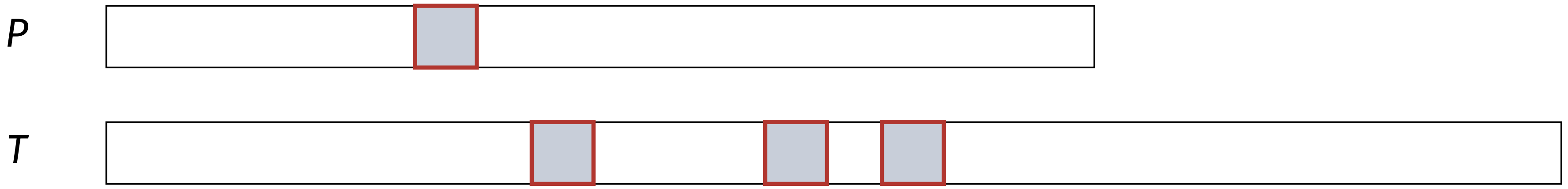
B has $O(k)$ occurrences in T since $n \leq {}^3/_2\, m$ and the period of B is $> m/(128k)$.



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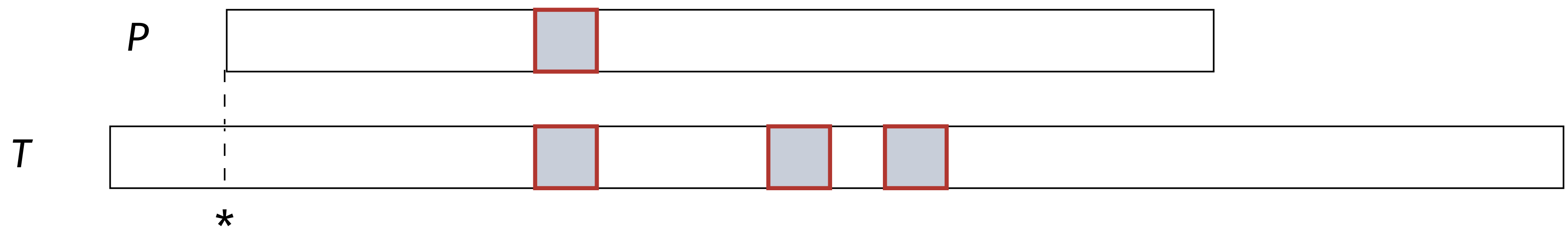
If P contains $\geq 2k$ disjoint breaks, there are $O(k)$ k-mismatch occ's of P in T.

For each such occurrence, we put a mark in the position of T where P starts if we align the break with the occurrence.



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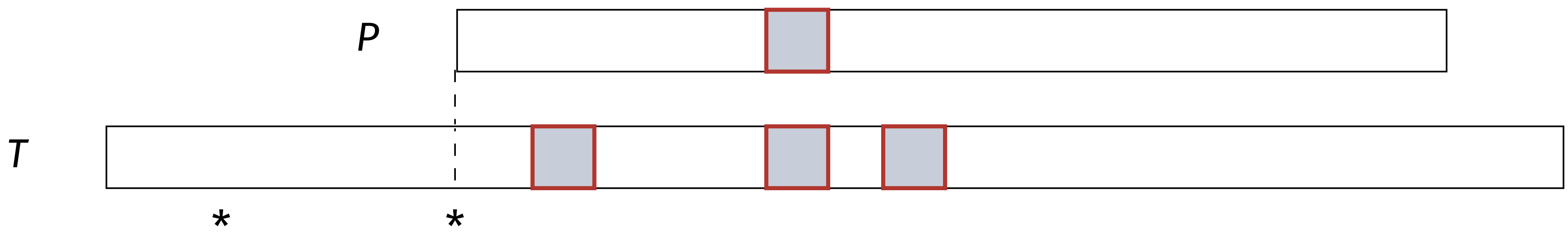

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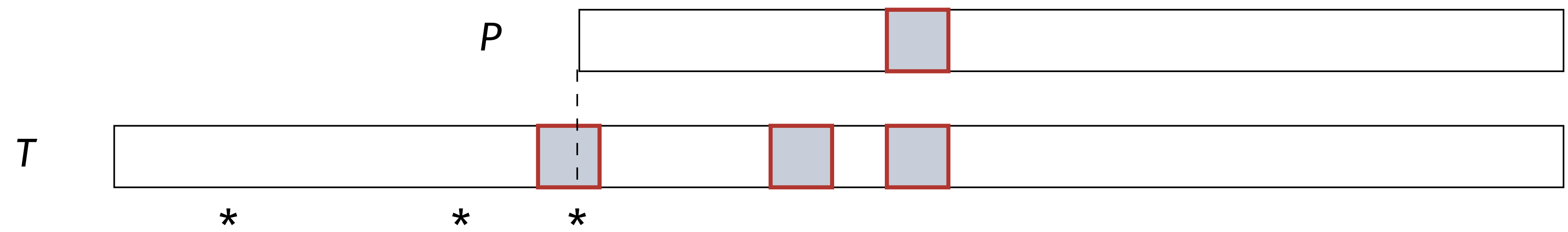

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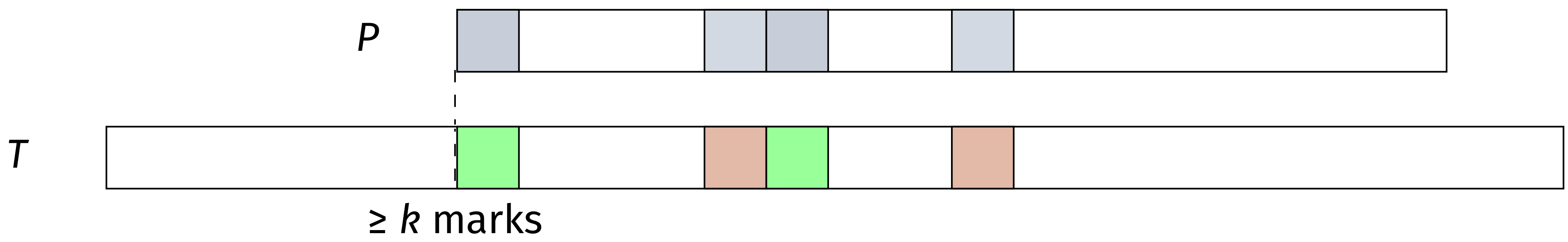
If P contains $\geq 2k$ disjoint breaks, there are $O(k)$ k-mismatch occ's of P in T.

Over all breaks, we place $O(k^2)$ marks. A position of T can be the starting position of a k-mismatch occurrence of P in T only if it has $\geq k$ marks.



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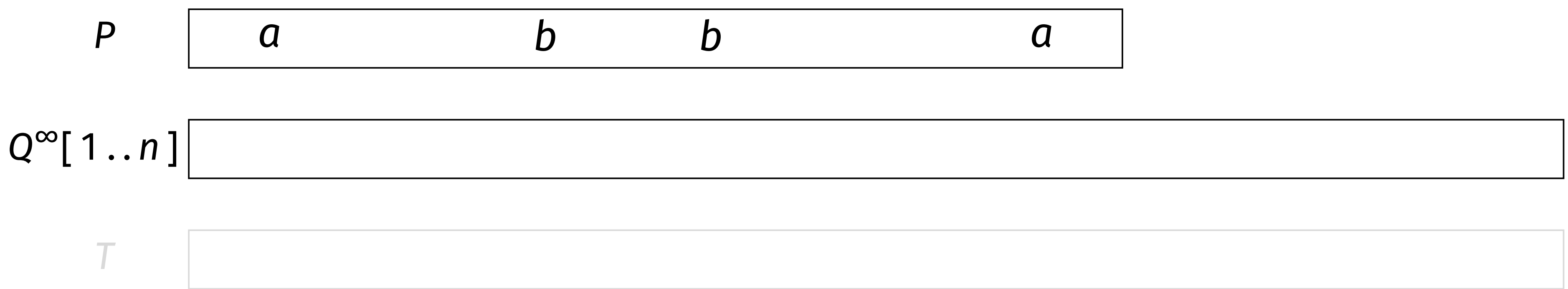
Consider again $k = 2$. We denote only the letters at positions where P differs from the length-m prefix of Q^∞, that is the misperiods.



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If P has HD $\geq 2k$ and $< 8k$ to a string w/ period $O(m/k)$, there are $O(k)$ k-mism. occ's of P in T.

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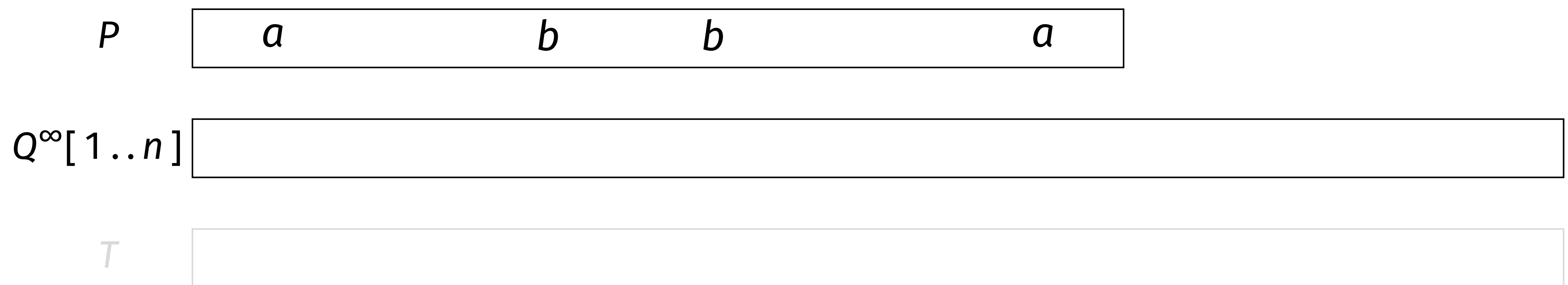

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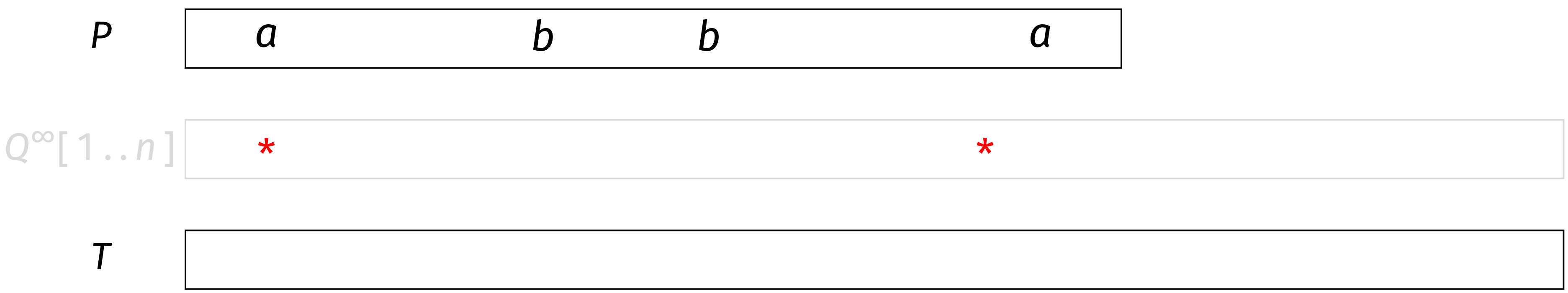

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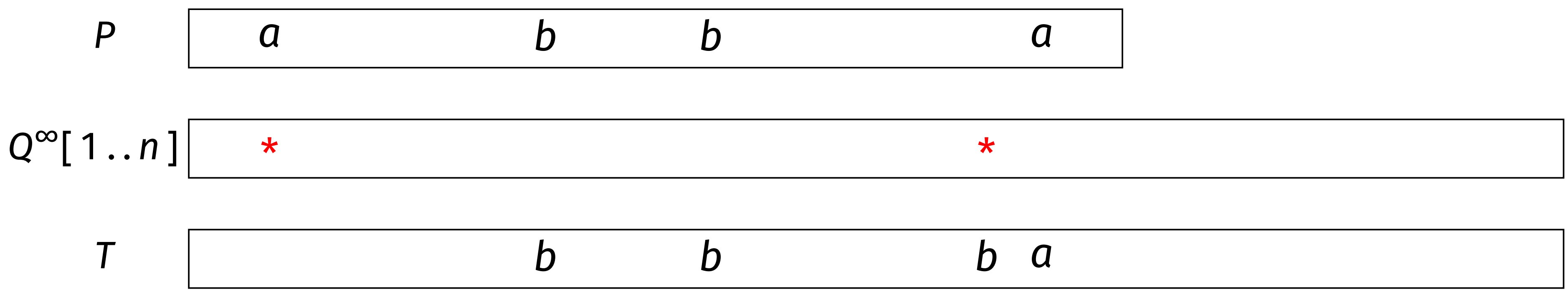

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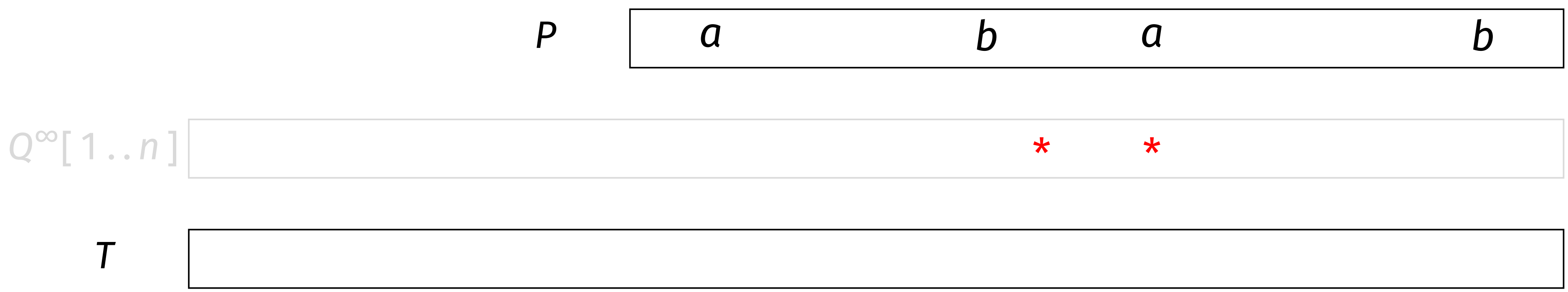

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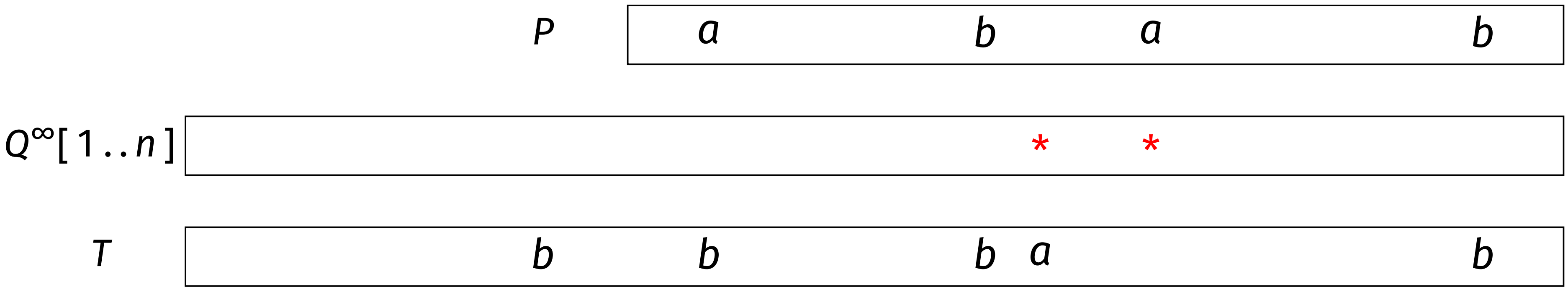

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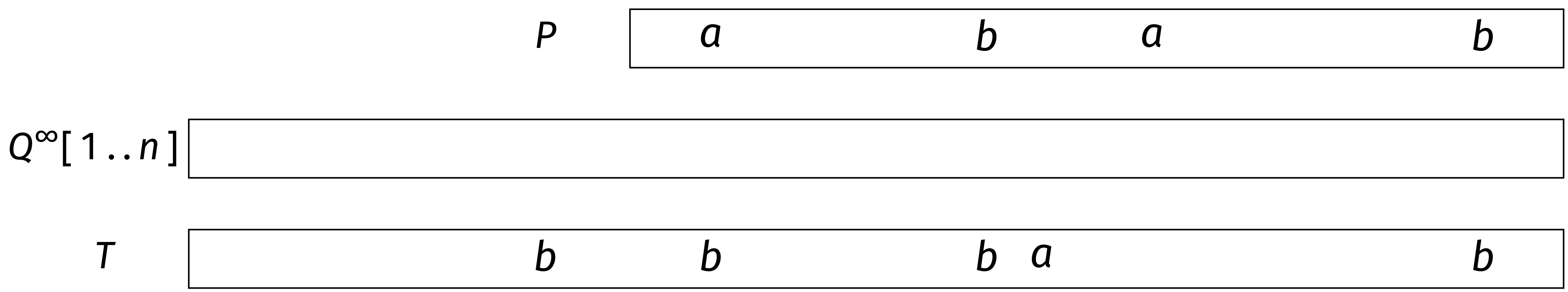
If P has HD $\geq 2k$ and $< 8k$ to a string w/ period $O(m/k)$, there are $O(k)$ k-mism. occ's of P in T.

P can only have k-mismatch occurrences in positions equiv. $1 \mod |Q|$ due to periodicity.
(We have many copies of Q, and Q does not match any of its rotations.)



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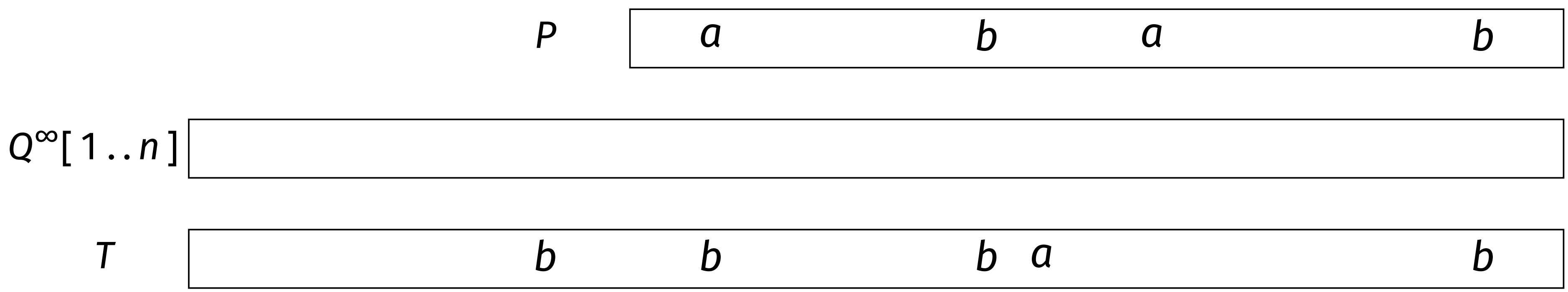
For each k-mismatch occurrence $T[i..j]$ of P, as P has $\geq 2k$ mismatches with $Q^\infty[1..m]$, at least k of P's misperiods must coincide with misperiods of $T[i..j]$.



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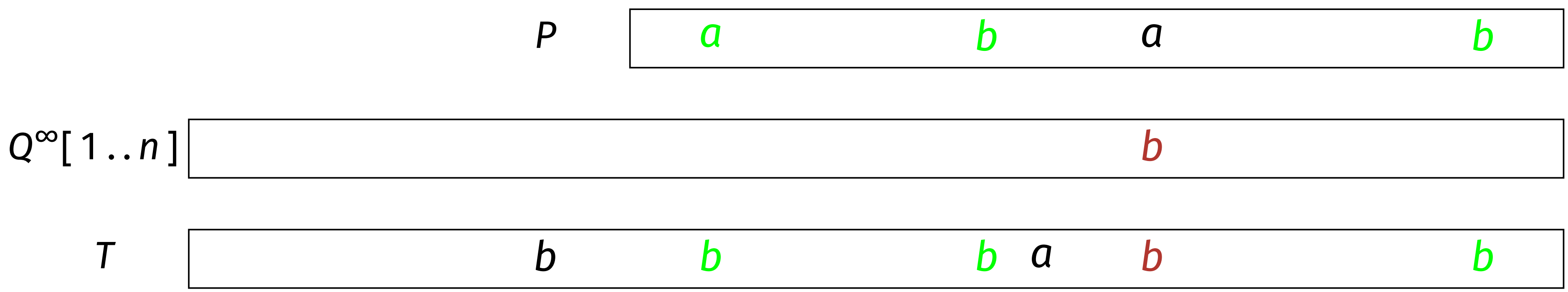

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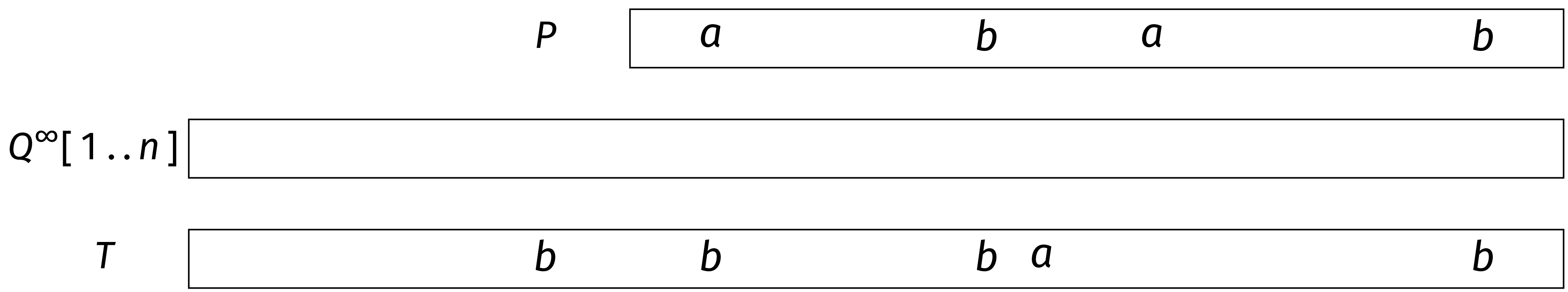
As we have $\Theta(k^2)$ pairs of misperiods, we have $O(k)$ k-mismatch occurrences of P in T.



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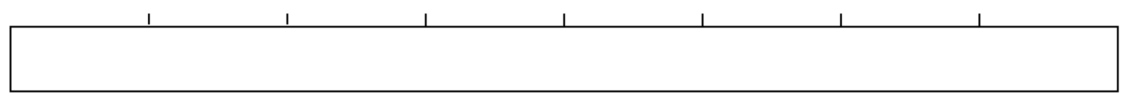
P



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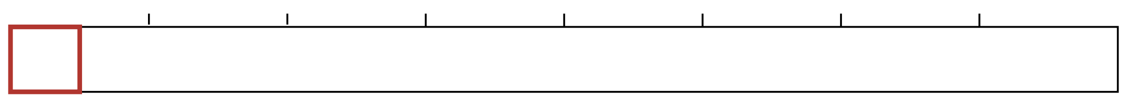
P 

- Process P from left to right, $m/8k$ new characters at a time.

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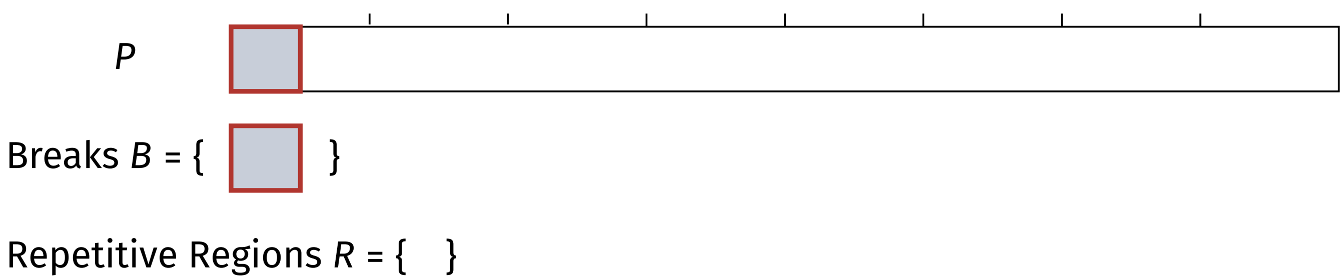

- If a fragment has a period $> m/128k$, add it to the found breaks.

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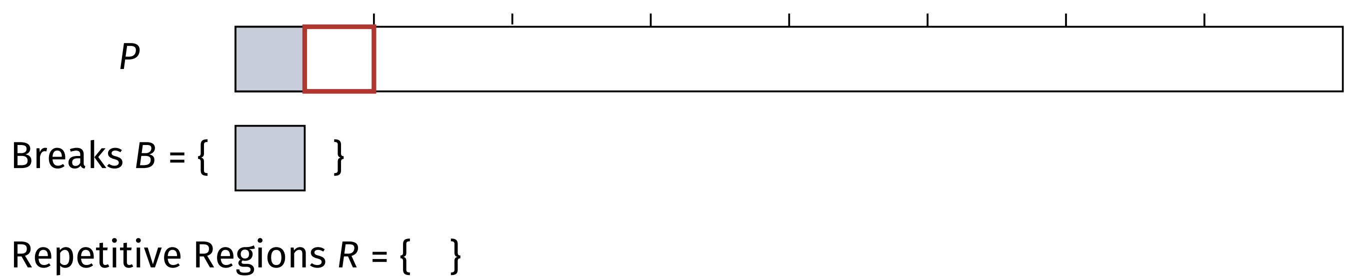

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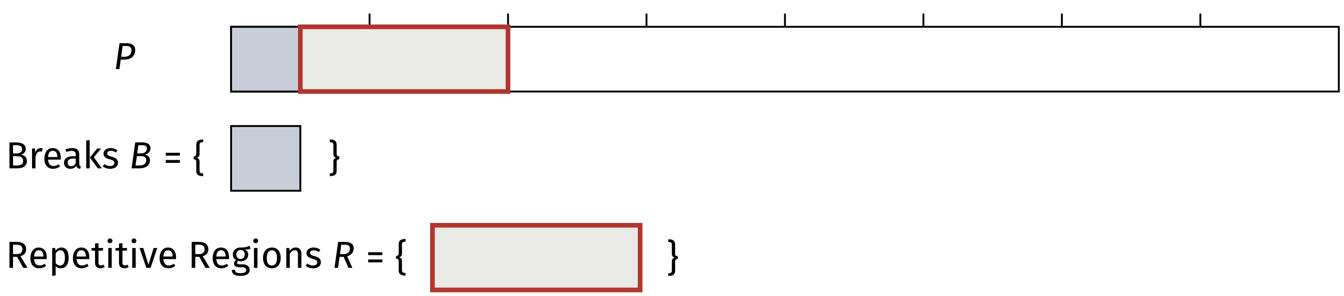

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- P contains $2k$ disjoint breaks; each break has length $m/8k$ and period $> m/(128k)$.
- P contains disjoint repetitive regions R_i with total length $\geq 3/8 \cdot m$; each region has length $\geq m/8k$ and is almost periodic with HD exactly $8k/m \cdot |R_i|$.
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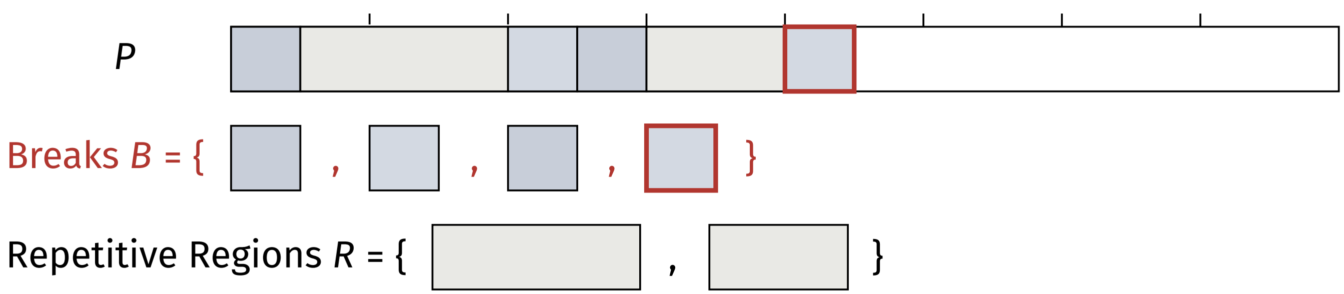

- If we found $2k$ breaks, return the breaks.

Analyzing the Pattern, Proof Idea

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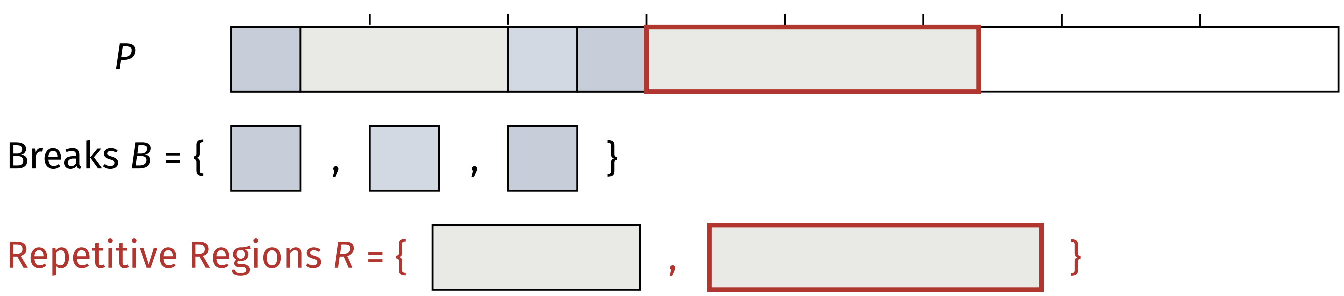

- If the total length of the repetitive regions is $> 3/8 \cdot m$, return the repetitive regions.

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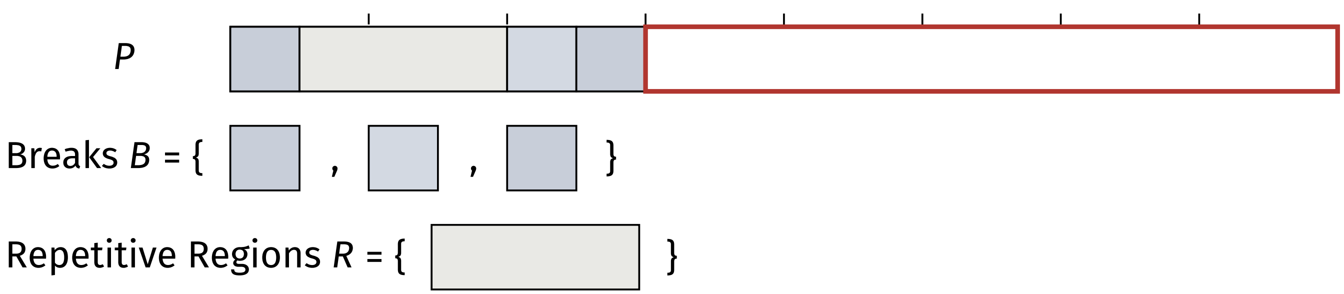

- If we reach the end of P, try to find a single repetitive region starting from the end.

Analyzing the Pattern, Proof Idea

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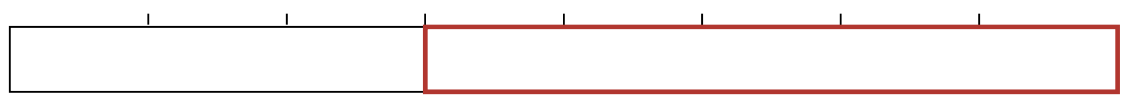
P



Breaks $B = \{ \quad \}$

Repetitive Regions $R = \{ \quad \}$

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Analyzing the Pattern, Proof Idea

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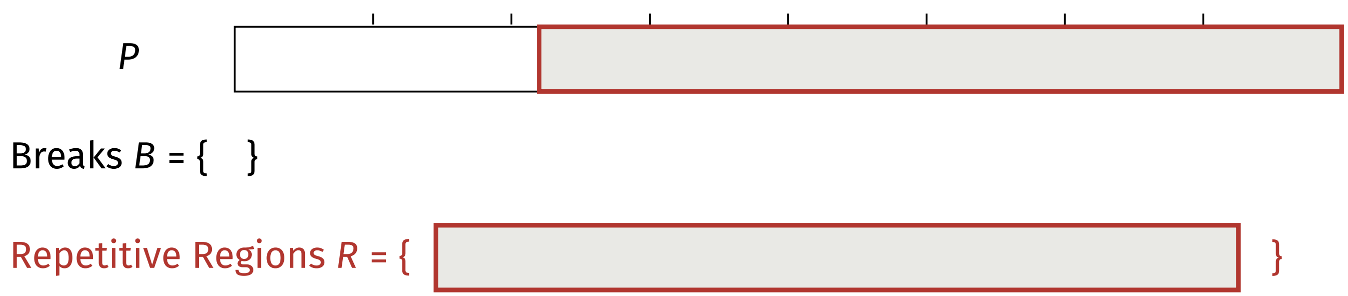

- If we found a repetitive region, return it.

Analyzing the Pattern, Proof Idea

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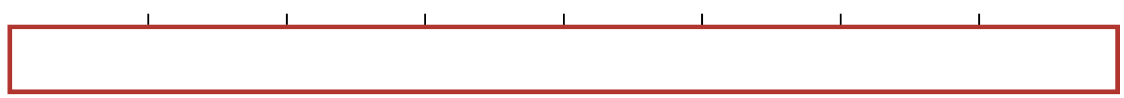

Breaks $B = \{\ \}$

Repetitive Regions $R = \{\ \}$

- If we again don't obtain a repetitive region, P is almost periodic.

Analyzing the Pattern, Proof Idea

Key Lemma (Analyze) ✓

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What about PM with errors?

Structural Results for PM with Errors

Structural Theorem (HD) [C.-Kociumaka-Wellnitz, FOCS 2020]

Given a pattern P of length m, a text T of length $n \le {}^{3}\!/_{2}\, m$, and a threshold $k \le m$, at least one of the following holds:

- The number of k-mismatch occurrences of P in T is $O(k)$.
- The pattern P is almost periodic (at HD < $2k$ to a string Q with period $O(m/k)$).

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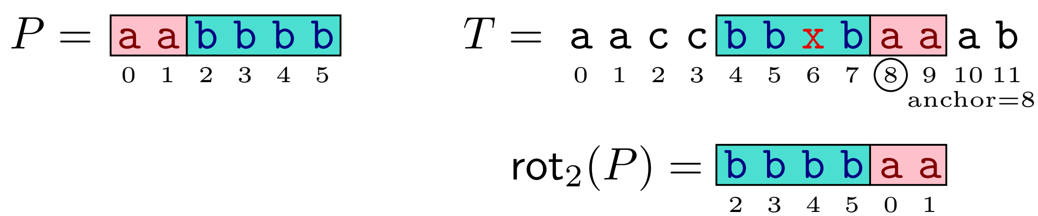
- The starting positions of all k-error occurrences of P in T lie in $O(k)$ intervals of length $O(k)$ each.
- The pattern P is almost periodic (at ED $< 2k$ to a string Q with period $O(m/k)$).

What next?

Consider more complicated settings.

For instance, the case where (approximately) matching any rotation of the pattern is acceptable has been already considered.



Either the pattern is almost periodic or there are $O(k)$ anchors for k-mismatch circular occurrences. (Each anchor gives $O(k)$ intervals of occurrences.)
[CKPRRSWZ, ESA'22]

Results on Approximate Pattern Matching under HD

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- $\tilde{O}(n + k^2 \cdot n/m)$ – [CKW'20] (improvement in log-factors)

How do we turn the structural insights into algorithms?

Obtaining Faster Algorithms

- Create algorithms that rely on a small set of essential operations:
  - `LCP`(S, T): Compute the length of the longest common prefix of S and T.
  - `LCP`^{R}(S, T): Compute the length of the longest common suffix of S and T.
  - `IPM`(P, T): Compute all exact matches of P in T.
  - `Length`(S): Compute the length $|S|$ of S.
  - `Access`(S, i): Retrieve the character $S[i]$.
  - `Extract`(S, ℓ, r): Extract the fragment (or substring) $S[\ell\,..\,r)$ from S.

The `PILLAR` Model

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Theorem
Fast PILLAR Algorithm

+

Theorem
Fast PILLAR Implementation

=

Theorem
Fast Algorithm

An algorithm for the almost periodic case

Consider a pattern P that is at Hamming distance $< 2k$ from a prefix of Q^∞, where Q is primitive and $|Q| \le m/(128k)$, and a string P that is at Hamming distance $< 6k$ from a prefix of Q^∞.

Suppose that we are given the $O(k)$ misperiods for each of P and T.

How fast can we compute a representation of k-mismatch occurrences of P in T?

Hint 1: The exact occurrences of $(\mathrm{ab})^{70}$ in $(\mathrm{ab})^{100}$ can be represented as a single arithmetic progression $\{1 + 2i : i \in [0, 30]\}$.

Hint 2: k-mismatch occurrences of P can only start at positions of T that are $\equiv 1 \bmod |Q|$.

What Changes for Edit Distance?

Brief discussion on the board.

The PILLAR Model: Fast PILLAR Algorithms

The PILLAR operations: LCP, LCP^R, IPM, Length, Access, Extract

Theorem (PILLAR Alg. for PM w/ Mism.)

Given a pattern P of length m, a text T of length n, and a positive threshold $k \le m$, we can compute (a representation of) all k-mismatch occurrences of P in T using $O(n/m \cdot k^2 \log\log k)$ time plus $O(n/m \cdot k^2)$ PILLAR operations.

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Theorem (PILLAR Alg. for PM w/ Errors)

Given a pattern P of length m, a text T of length n, and a positive threshold $k \le m$, we can compute (a representation of) all k-error occurrences of P in T using $\tilde{O}(n/m \cdot k^{3.5})$ PILLAR operations.

The PILLAR Model: The Standard Setting (HD)

The PILLAR operations: LCP, LCP^R, IPM, Length, Access, Extract
Uncompressed strings: pattern P of length m, text T of length n.

The $\texttt{PILLAR}$ Model: The Standard Setting (HD)

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We can perform each operation in $O(1)$ time. (After $O(n + m)$-time preprocessing.)

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Theorem (Algorithm for PM w/ Mism.)

For any positive threshold $k \le m$,
we can compute all k-mismatch occurrences of P in T in time $O(n + n/m \cdot k^2 \log\log k)$.

What if the text and the pattern are *huge*?

o	f	s	w	e	e	t	b	e	e	n	c	e	n	t	e	r	e	d	i	n	t	h	e	a	n	p	a	n	f	o	r	e	x	a	m

n	p	a	s	t	e	a	n	p	a	n	c	a	n	a	l	s	o	b	e	p	r	e	p	a	r	e	d	w	i	t	h	o	t	h	e

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anpanisajapanesesweetrollmostcommonlyfilledwithredbeanpasteanpancanalsobepreparedwithotherfillingsincludingwhitebeansgreenbeanssesameandchestnut

What if the text and the pattern are given in a compressed representation?

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Grammar Compression

Grammar Compression

For a string T, a grammar compression of T is a context-free grammar G_T that generates $\{T\}$. The grammar G_T is wlog. a straight-line program or SLP.

Grammar Compression

Straight-Line Program (SLP)

An SLP G_T is a set of non-terminals $\{T_1, \dots, T_n\}$ and productions of the form $T_i \rightarrow a, a \in \Sigma$ or $T_i \rightarrow T_\ell T_r$, where $\ell, r < i$. The starting symbol is T_n.

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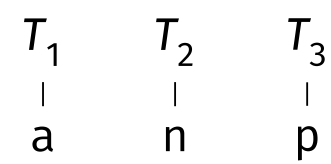

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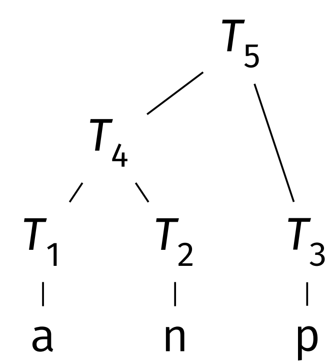
$$T_4 \rightarrow T_1 T_2; \qquad T_5 \rightarrow T_4 T_3$$



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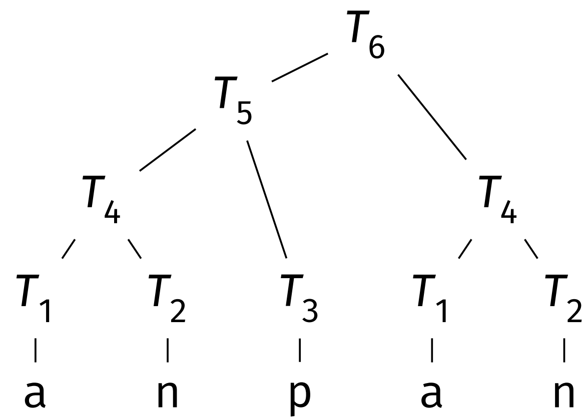

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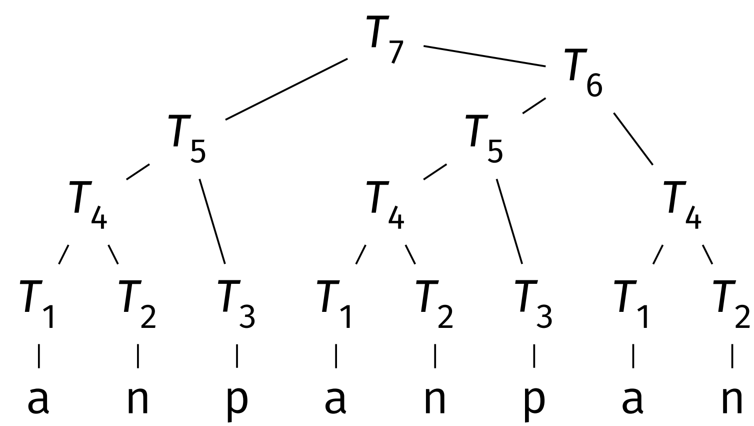

Faster Algorithms

Problem	uncompressed text and pattern	SLP text and pattern $n = \Omega(\log N)$, $m = \Omega(\log M)$
Pattern Matching	$O(N + M)$ [KMP'77]	$\tilde{O}(n + m)$ [Jeż'15]
PM with k Mismatches	$\tilde{O}(N + k^2 \cdot N/M)$, $\tilde{O}(N + kN/\sqrt{M})$ [CFPSS'16] [GU'18]	$\tilde{O}(nk^4 + Mk)$ [BKW'19]
PM with k Errors	$O(N + k^4 \cdot N/M)$ [CH'02]	$O(nm\,\mathrm{poly}(k))$ [BLRSSW'15]

N: length of uncompressed text

n: size of compressed text

k: number of mismatches/errors

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PM with k Errors	$O(N + k^4 \cdot N/M)$ [CH'02]	$\cancel{O(nm\,\mathrm{poly}(k))}$ $\tilde{O}(nk^{3.5} + m)$

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Faster Algorithms

Problem	uncompressed text and pattern	SLP text and pattern $n = \Omega(\log N)$, $m = \Omega(\log M)$
Pattern Matching	$O(N + M)$ [KMP'77]	$\tilde{O}(n + m)$ [Jeż'15]
PM with k Mismatches	$\tilde{O}(N + k^2 \cdot N/M)$, $\tilde{O}(N + kN/\sqrt{M})$ $\tilde{O}(N + k^2 \cdot N/M)$	$\cancel{\tilde{O}(nk^4 + Mk)}$ $\tilde{O}(nk^2 + m)$
PM with k Errors	$O(N + k^4 \cdot N/M)$ $\tilde{O}(N + k^{3.5} \cdot N/M)$	$\cancel{O(nm\,\mathrm{poly}(k))}$ $\tilde{O}(nk^{3.5} + m)$

N: length of uncompressed text

M: length of uncompressed pattern

n: size of compressed text

m: size of compressed pattern

k: number of mismatches/errors

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Improvements obtained via improved/new structural insights in solution structure.

Known Results: The Fully-Compressed Setting (HD)

The `PILLAR` operations: `LCP`, $\texttt{LCP}^R$, `IPM`, `Length`, `Access`, `Extract`

SLPs: G_P of size m generating pattern P, G_T of size n generating text T.

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For any positive threshold $k \le |P|$, we can compute the number of all k-mismatch occ's of P in T in time $O(m \log(|P| + |T|) + nk^2 \log^3(|P| + |T|))$.

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Let us see how on the board!

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Dynamic maintenance of a collection of (non-empty persistent) strings X of total length N; supporting `makestring`, `concat`, `split`.

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Theorem (Algorithm for PM w/ Mism.)

For any two strings $P, T \in X$ and any threshold k, we support the additional operation "Find all k-mismatch occ's of P in T" in $O(|T|/|P| \cdot k^2 \log^2 N)$ time (w.h.p).

Known Results: The Quantum Setting

Exercise

`PILLAR` operations can be performed in roughly $O(\sqrt{n})$ time by a quantum computer (without any preprocessing). How fast can we solve approximate pattern matching under each of the two studied distances in the quantum setting using the results we have already seen?

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For HD, an $\tilde{O}(k\sqrt{n})$-time algorithm is known [Jin-Nogler, SODA 2023].

Longest common extension (LCE) queries (standard setting)

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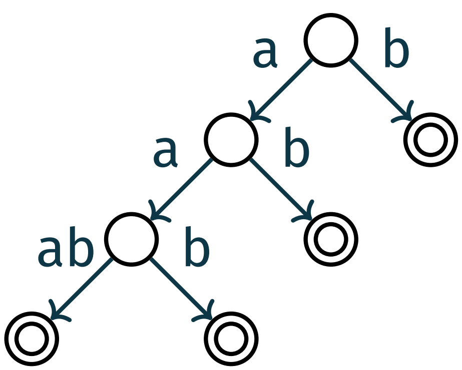

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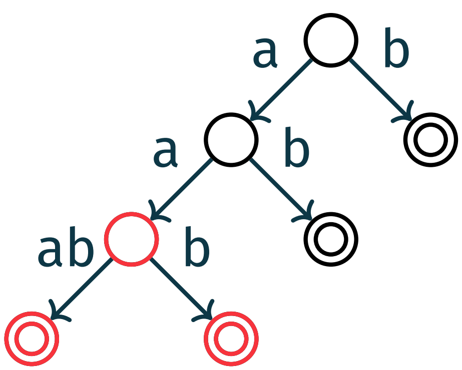

The longest common prefix of $S[1\,..\,4]$ and $S[2\,..\,4]$ is the string spelled on the path from the root to the lowest common ancestor of the two nodes "representing" these substrings.

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We will see a very cool reduction due to Mäkinen and Navarro [LATIN 2006].

IPM queries $\rightarrow$ 2D Range Reporting

T:	a	d	a	a	a	a	b	a	a	b	b	a	a	c
	1	2	3	4	5	6	7	8	9	10	11	12	13	14

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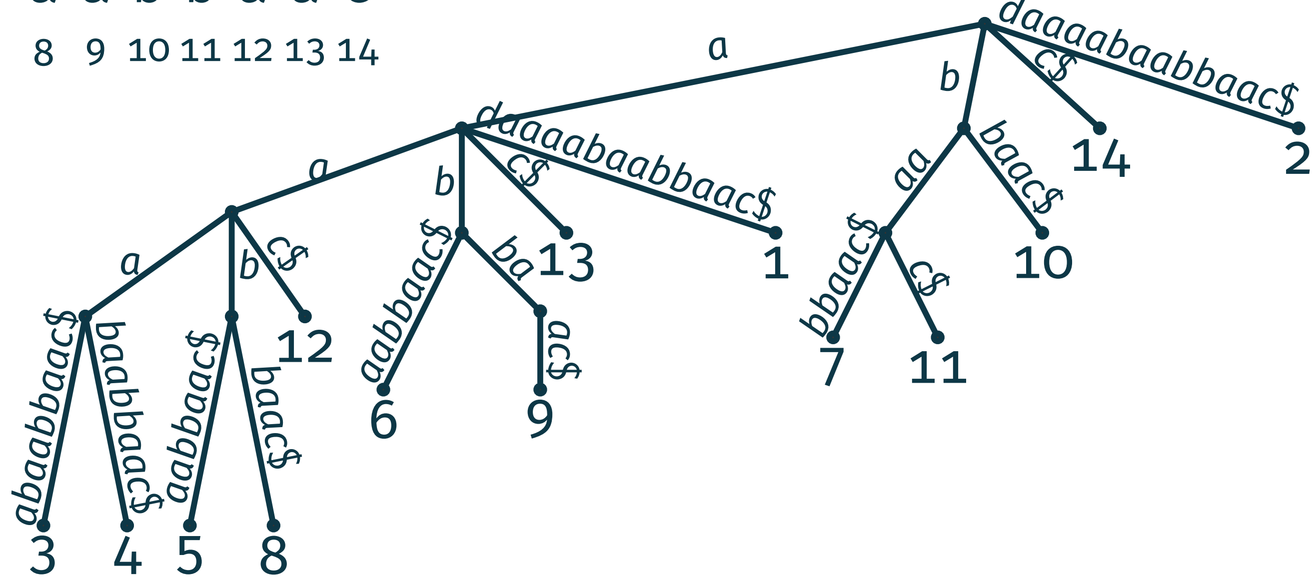

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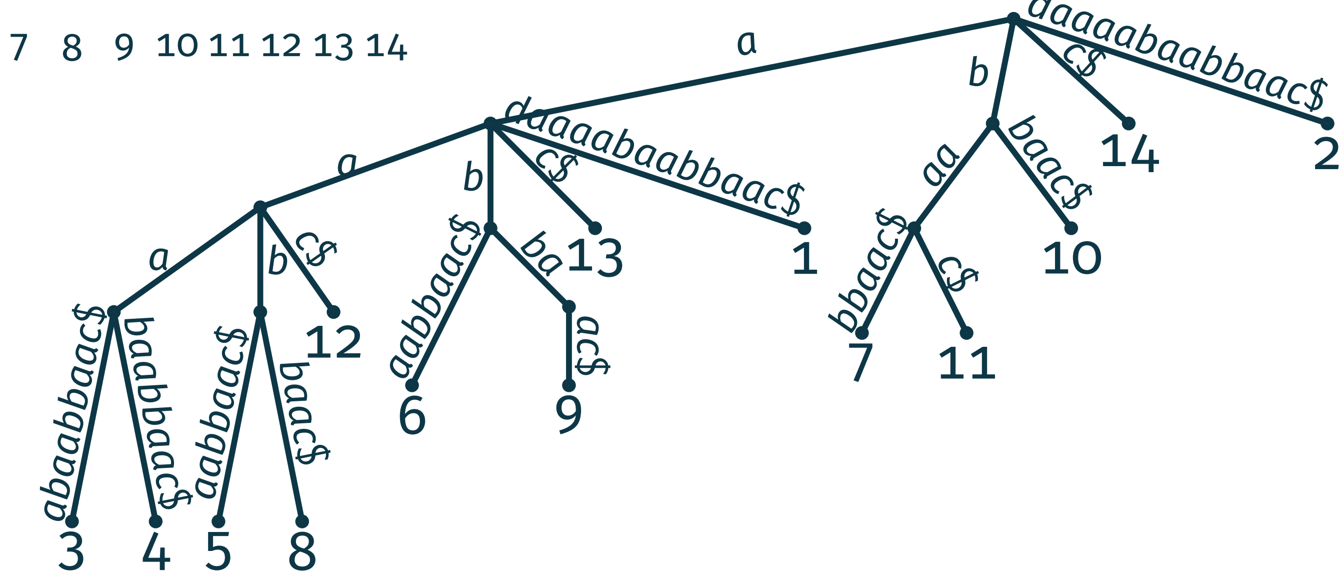

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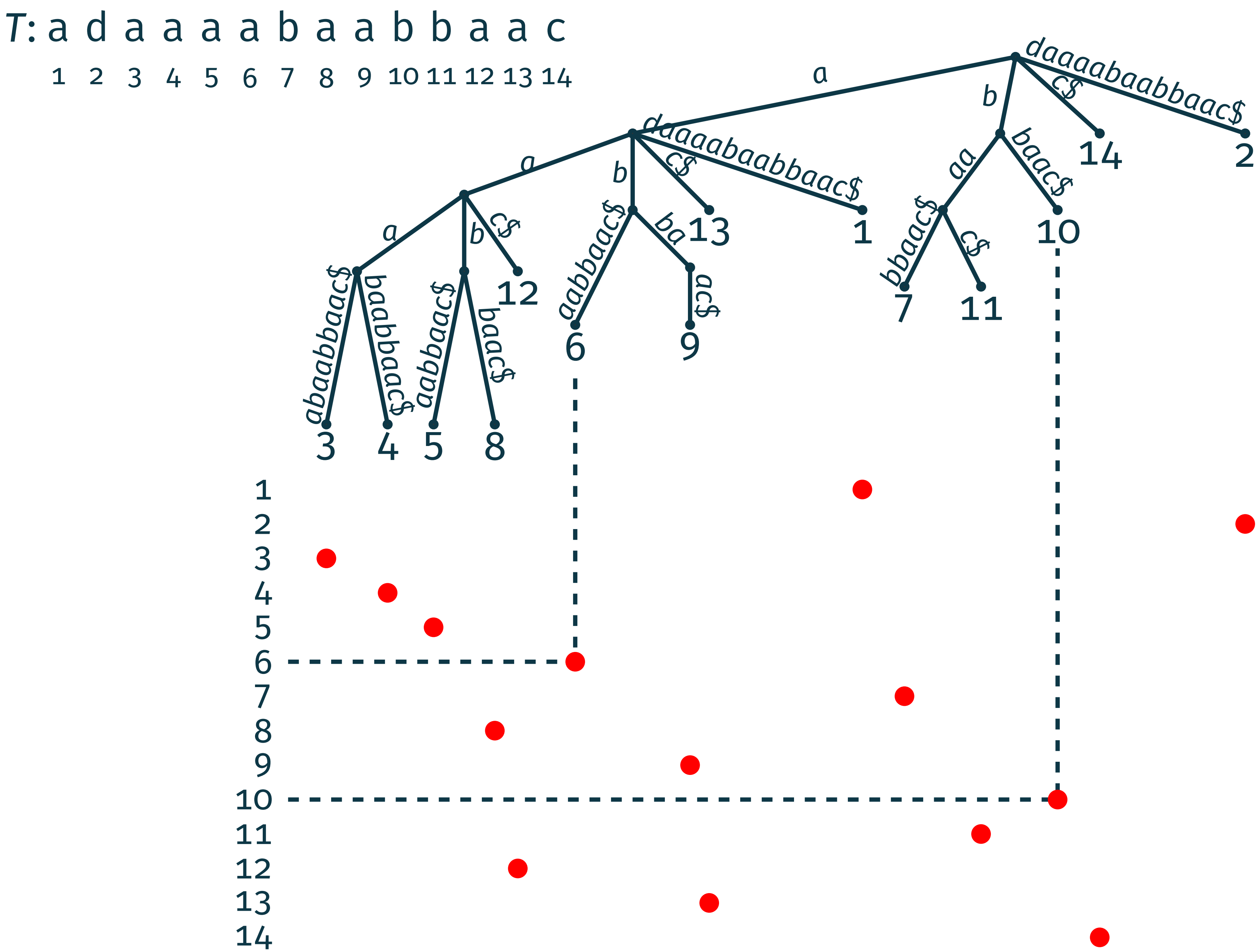

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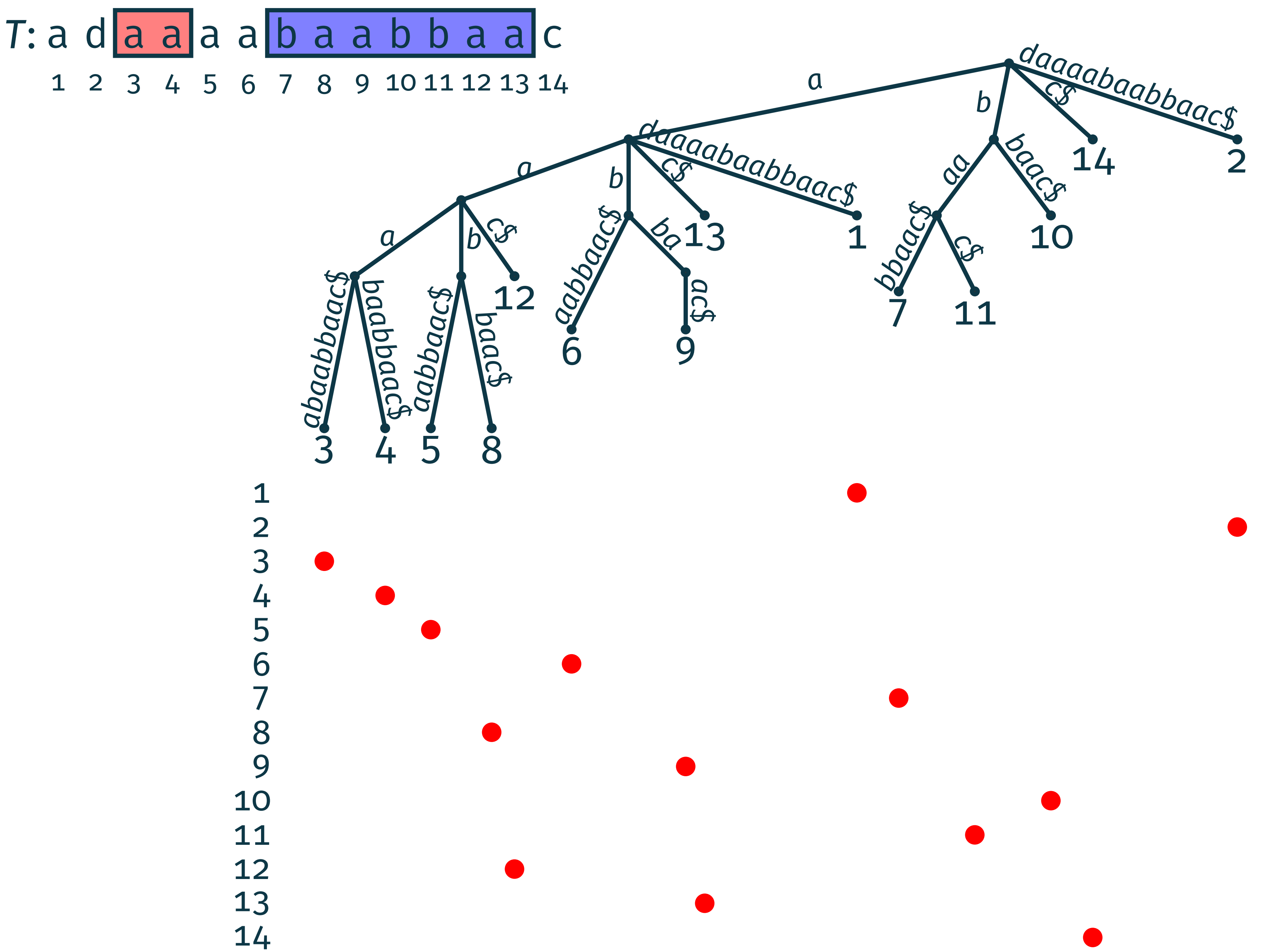

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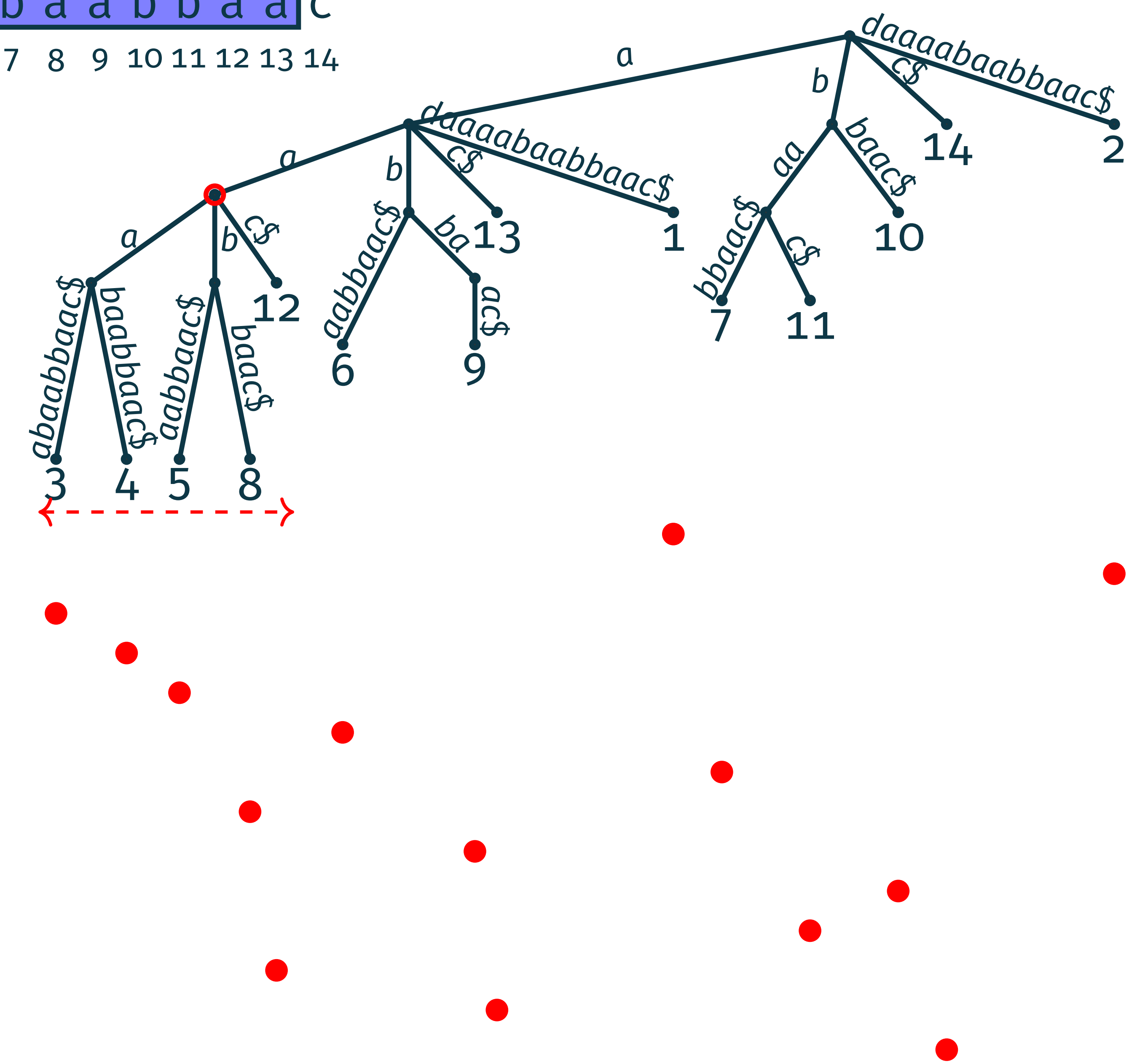

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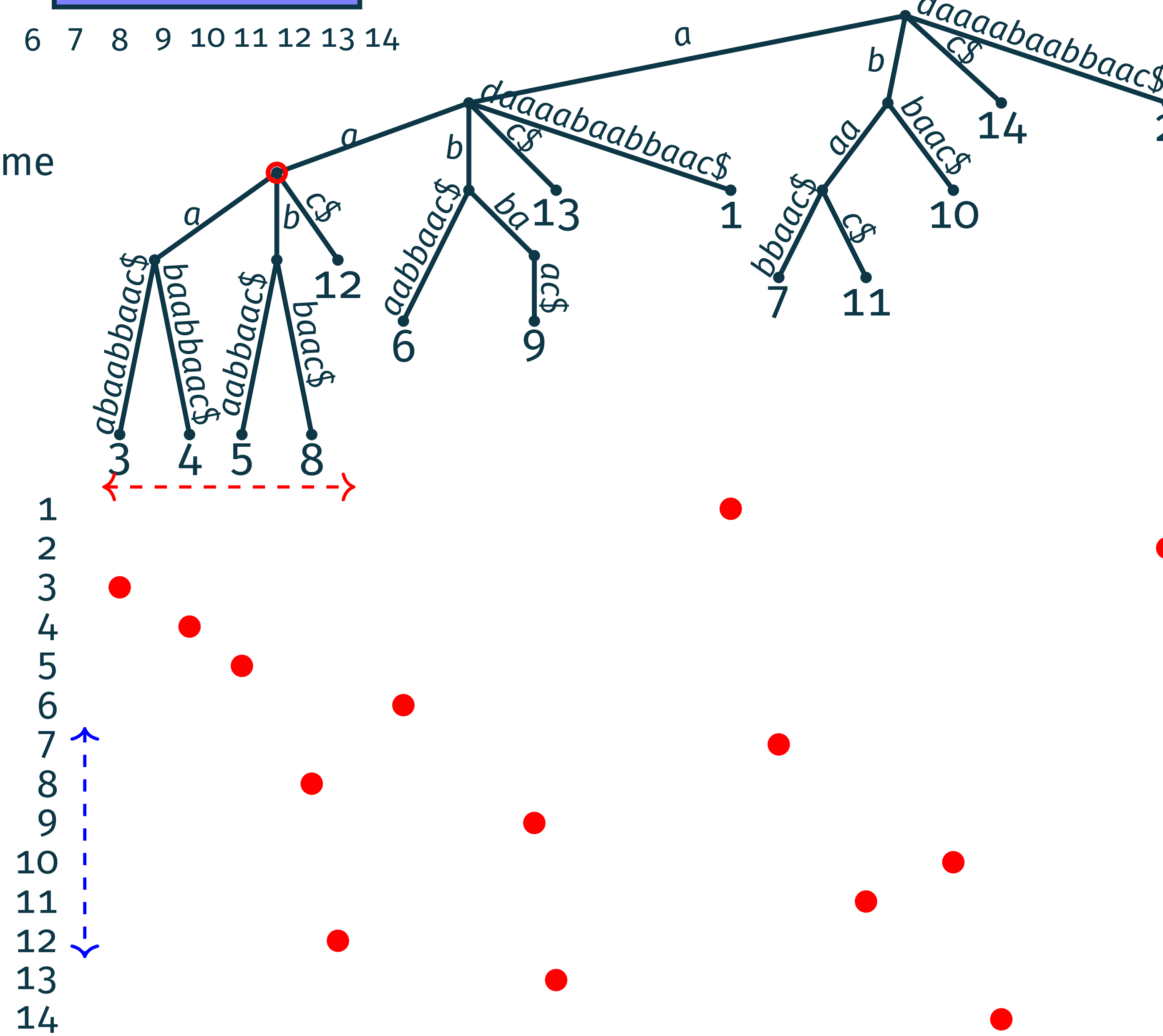

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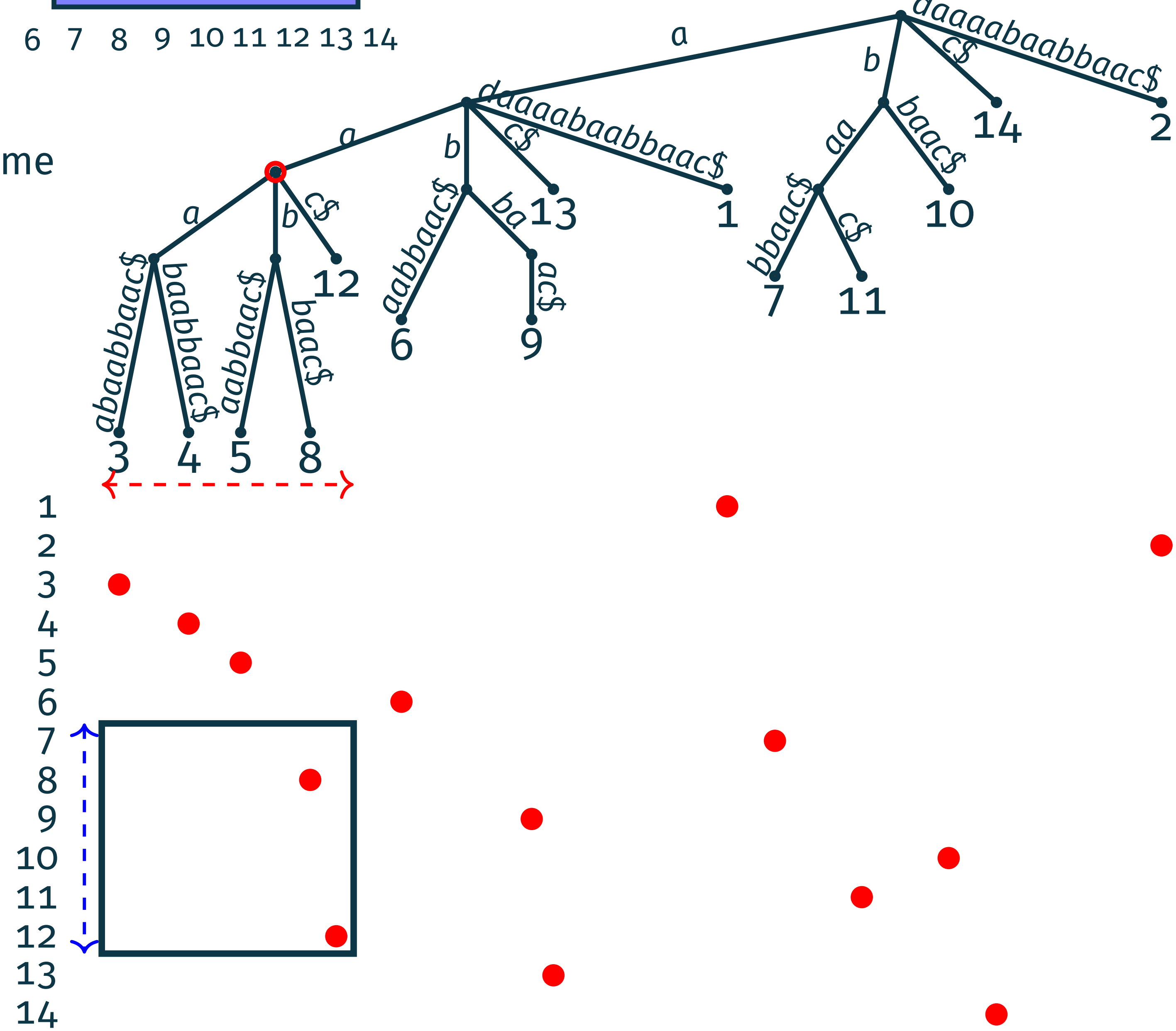

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Exercise: Reduce a query that checks if a fragment of a string S is periodic (and if so, also returns its period) to an internal pattern matching query and a longest common extension query on S.

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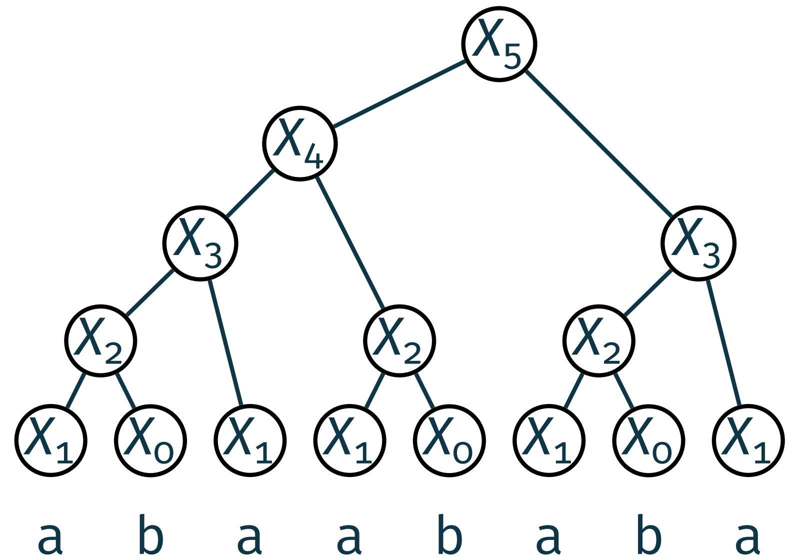

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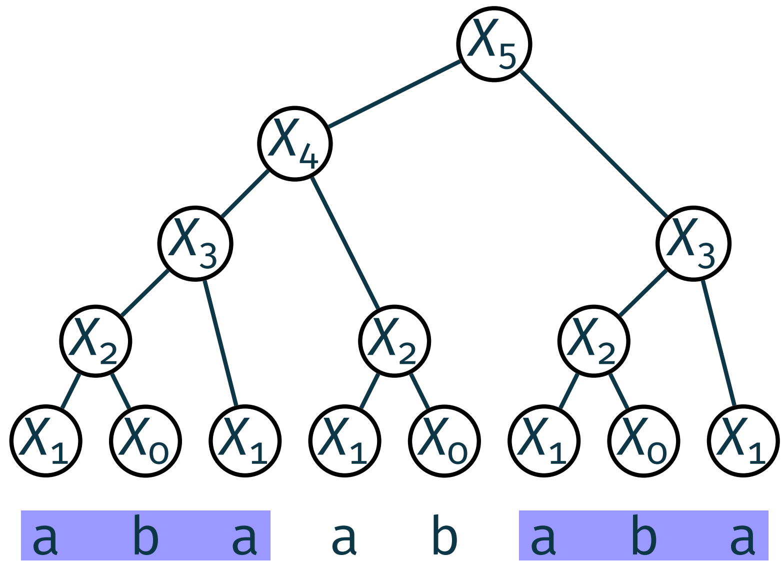

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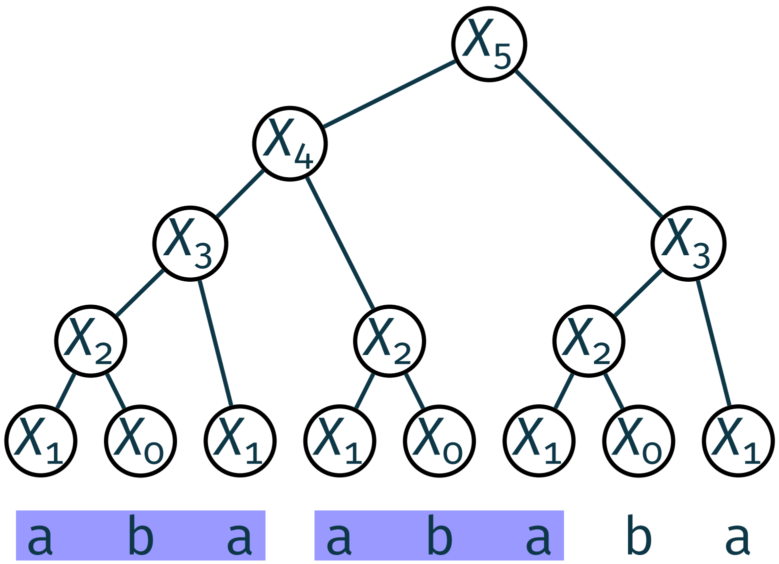

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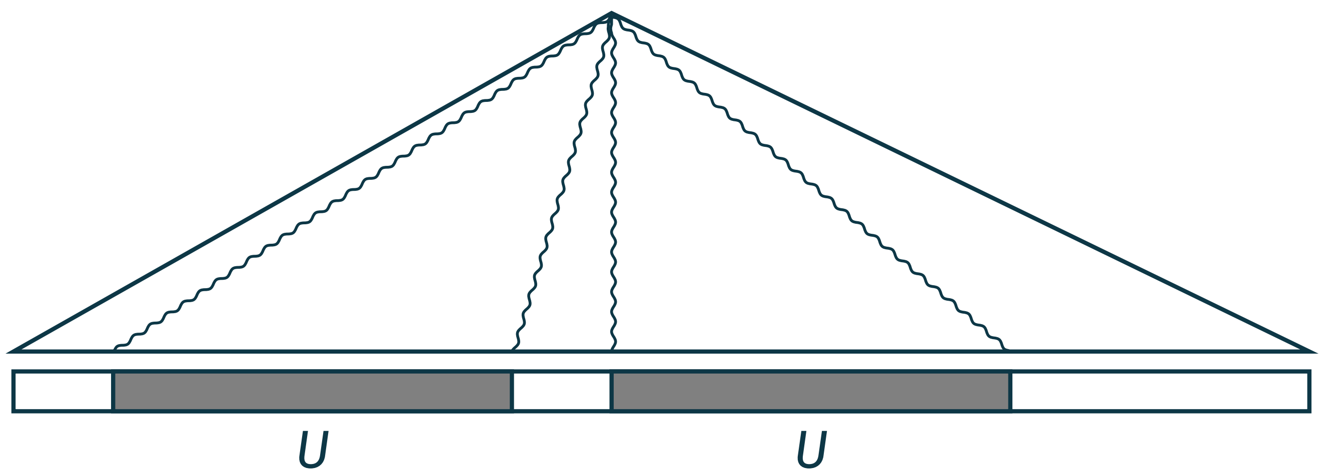

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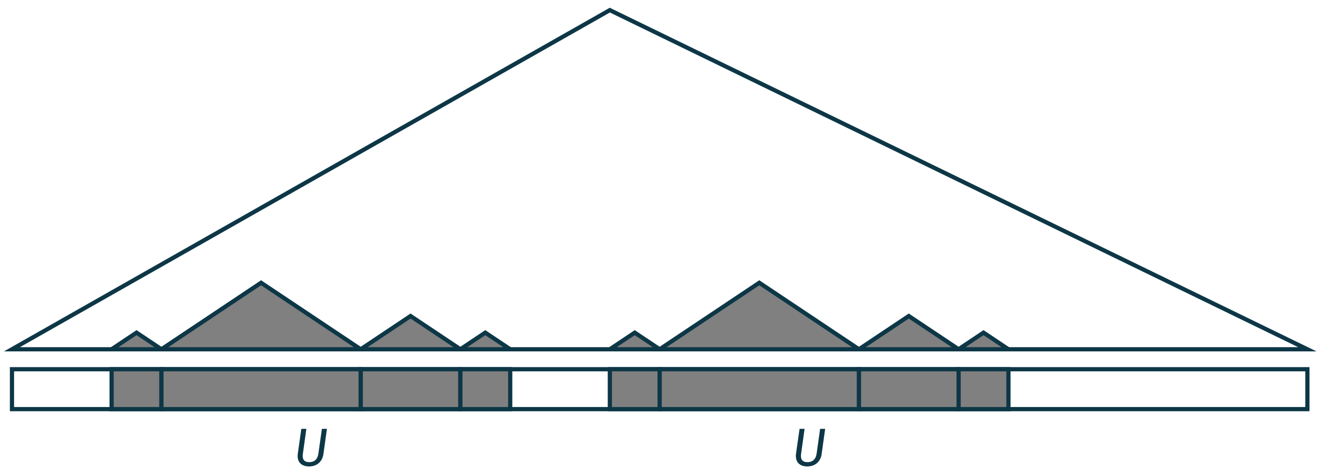

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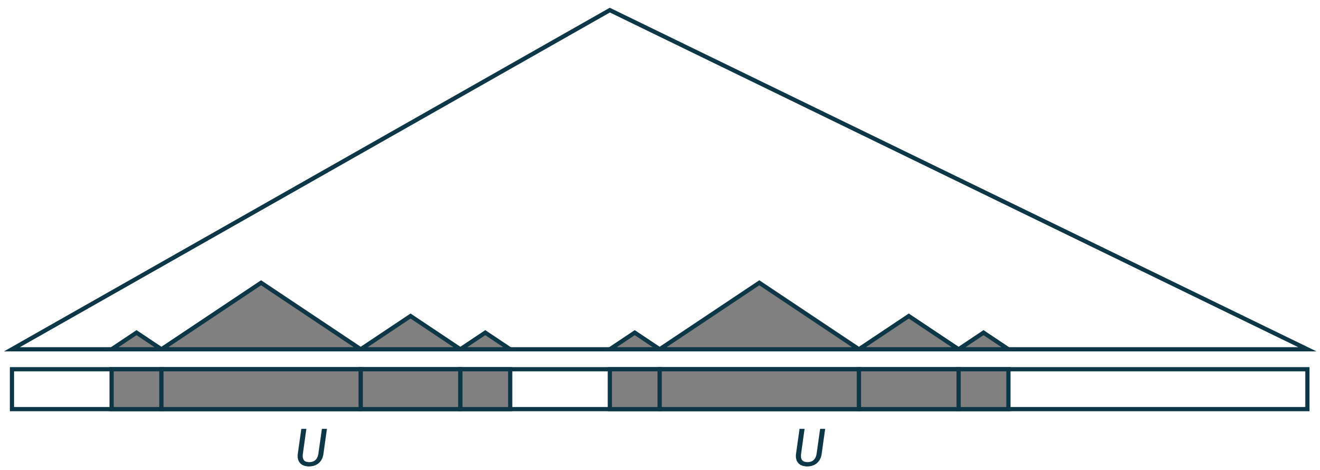

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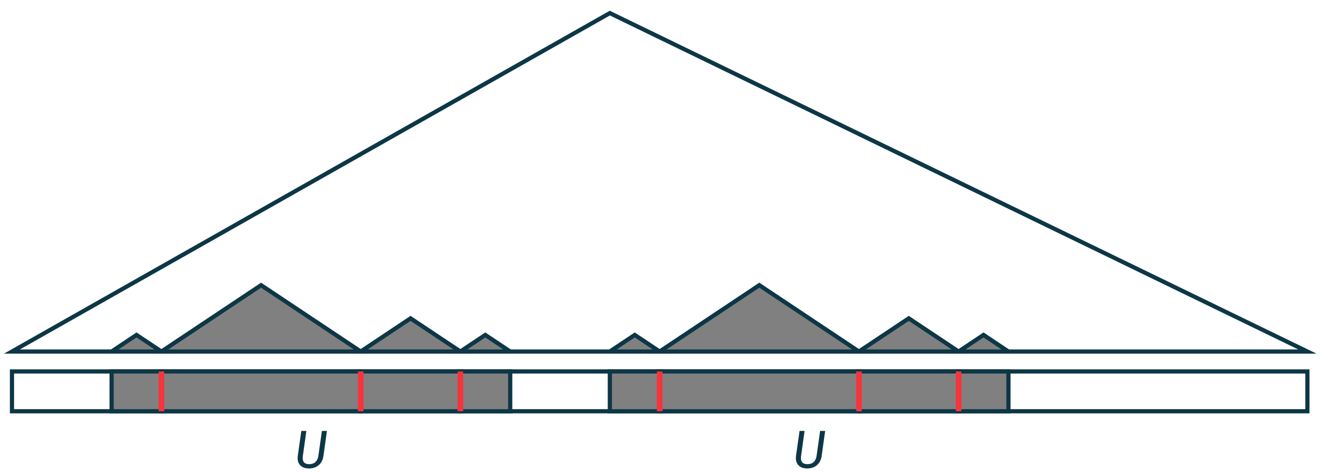

- Some nodes in the parse tree of w are preserved no matter the context.
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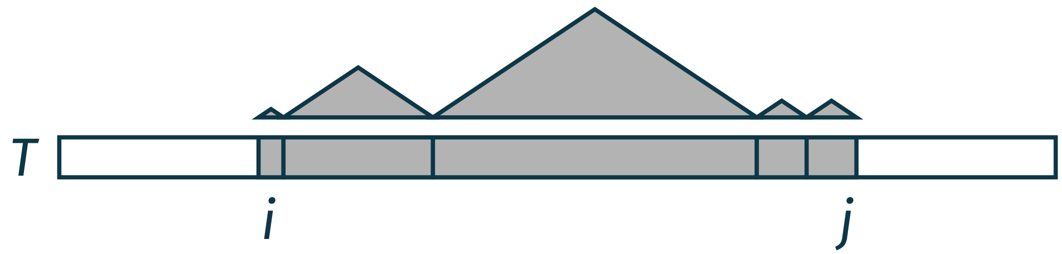
- These nodes define $\mathcal{O}(\log N)$ breakpoints for each substring; they partition it into phrases.

Occurrences of P in T



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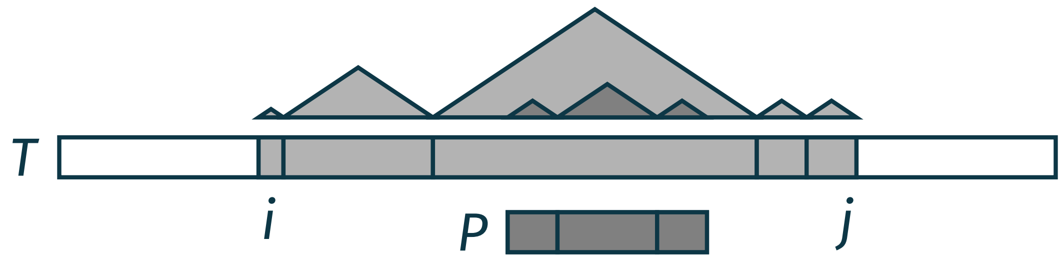

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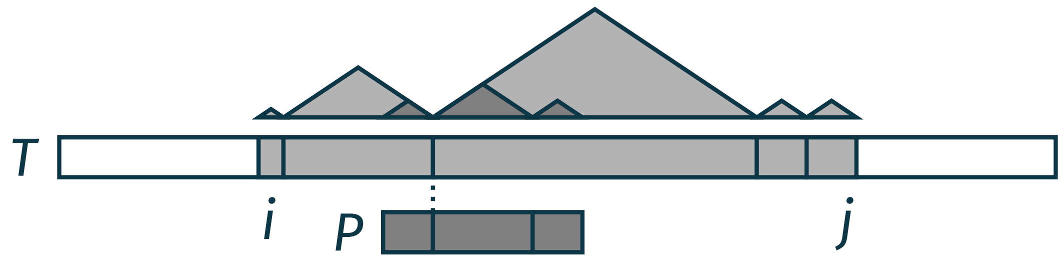

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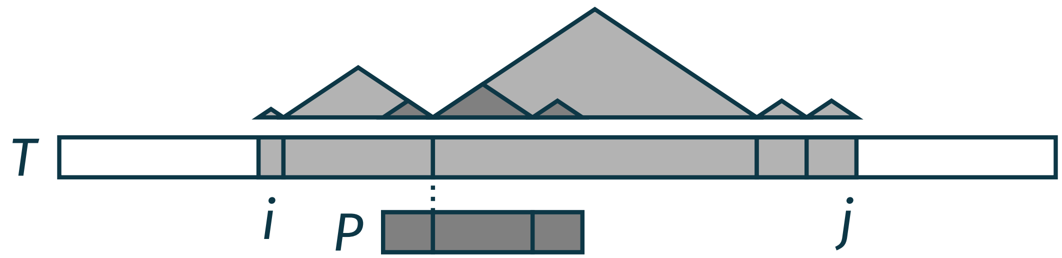

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Two SLPs that correspond to P and T, can be efficiently recompressed, so that the resulting parsings are locally consistent and have depth $\mathcal{O}(\log N)$.
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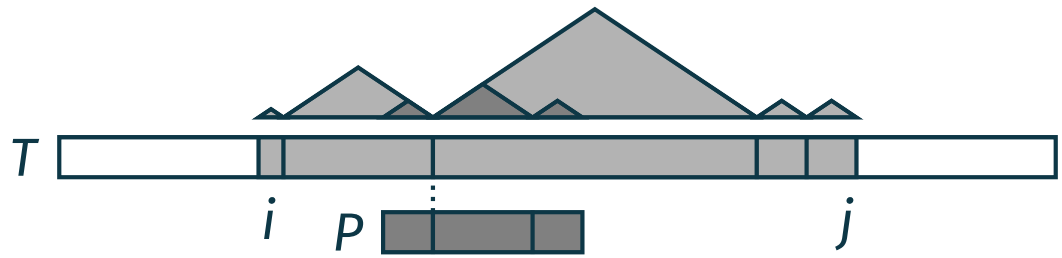

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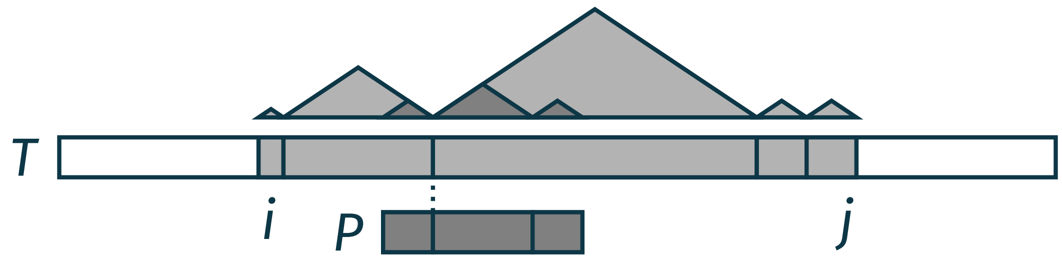

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How can we efficiently compute the occurrences of a substring U in a substring V if $|V| < 2|U|$? For each non-terminal in the parse tree whose production "breaks" a fragment of V of length at least m, try to align the breakpoint with each of U's $\mathcal{O}(\log N)$ breakpoints!

Recompression

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Run-length straight line program: context-free grammar generating exactly one string T by concatenations $A \to BC$ and powers $A \to B^k$.

Perform two types of steps interleaved until $|T| = 1$:

ⓒ ⓐ ⓒ ⓐ ⓑ ⓓ ⓑ ⓑ

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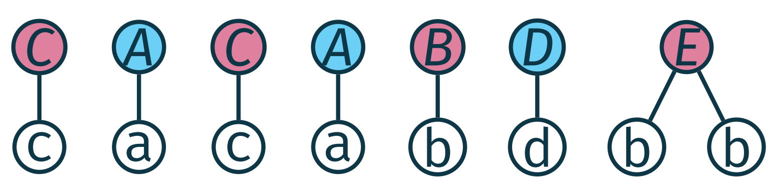

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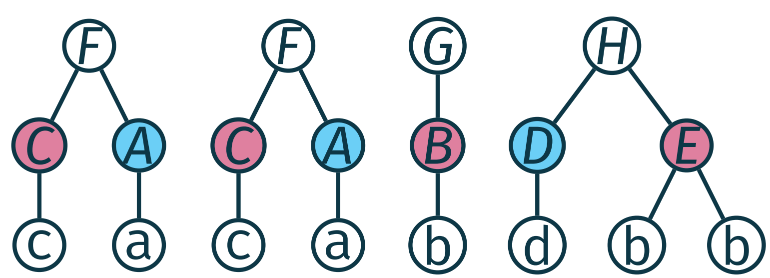

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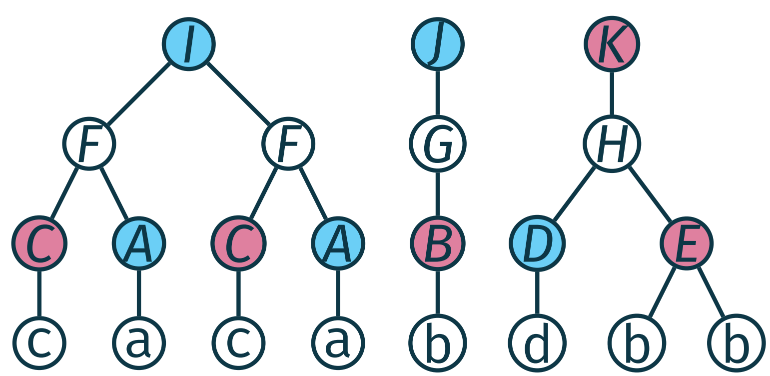

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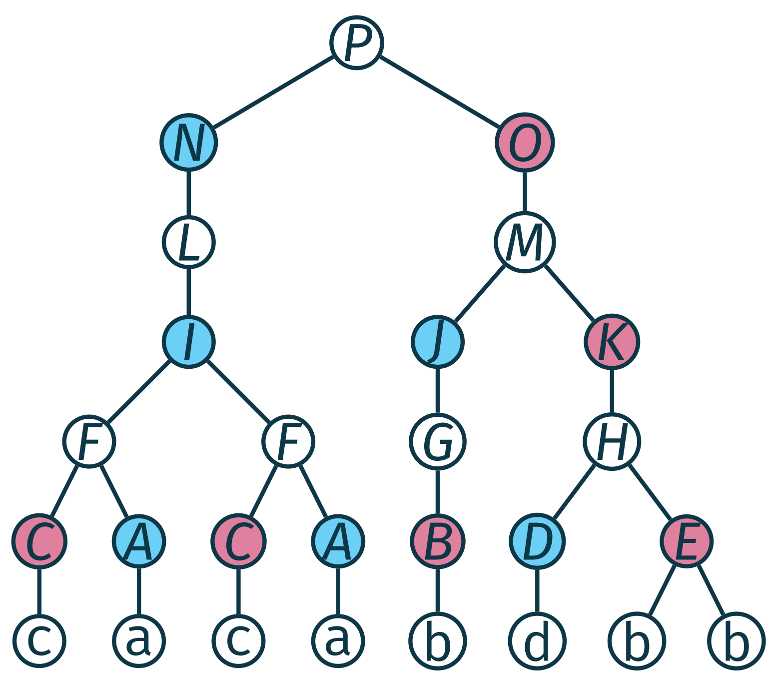

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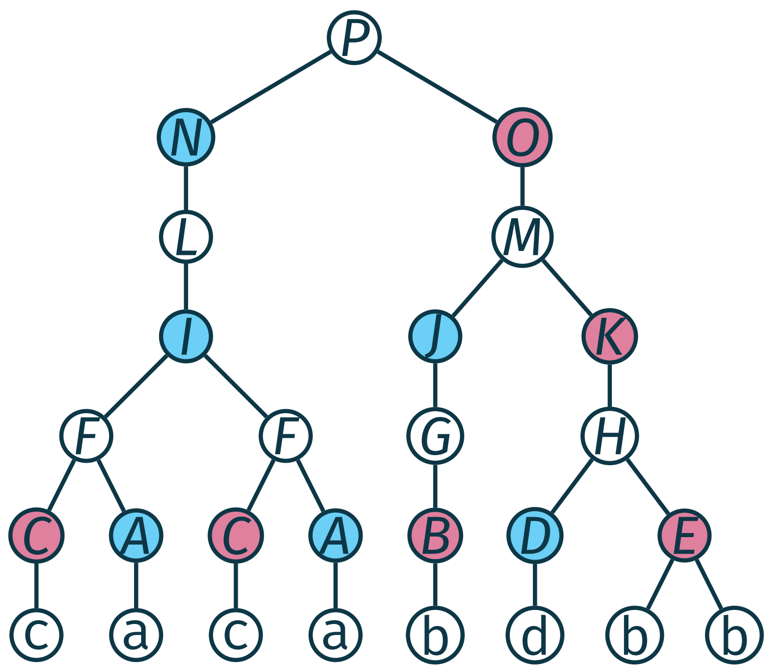

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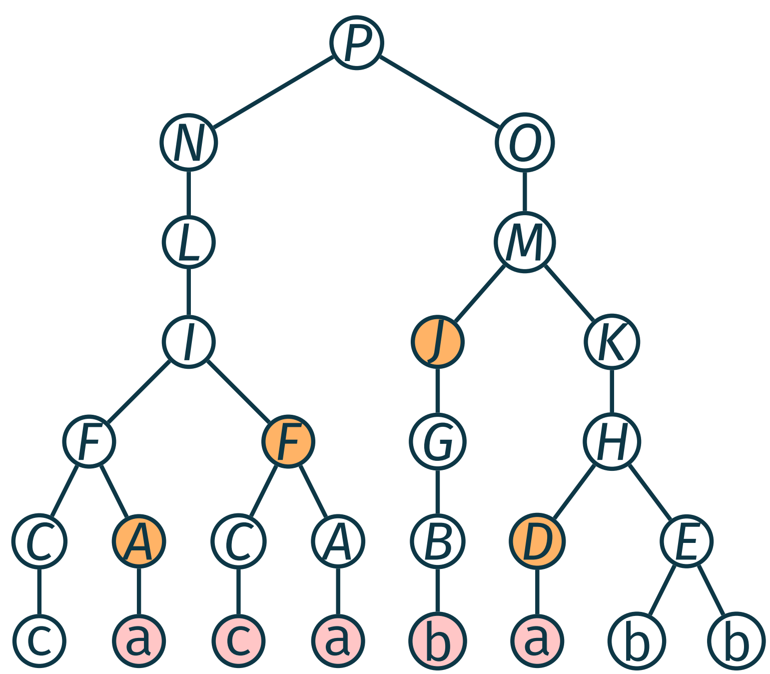

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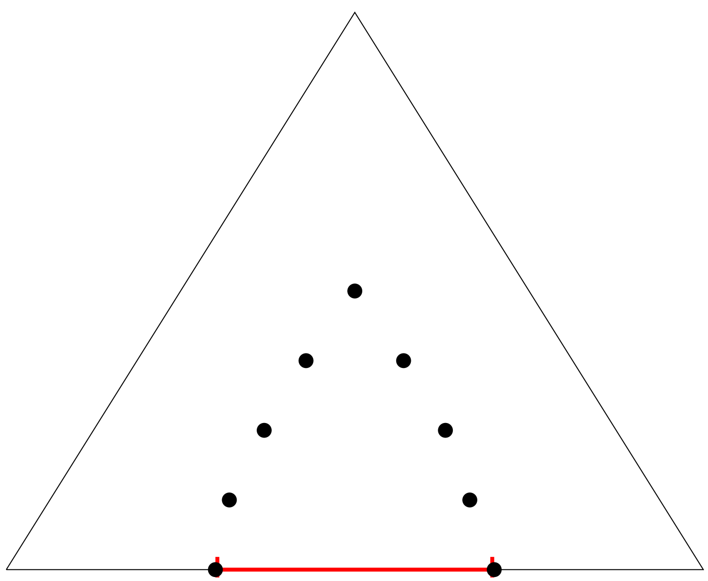

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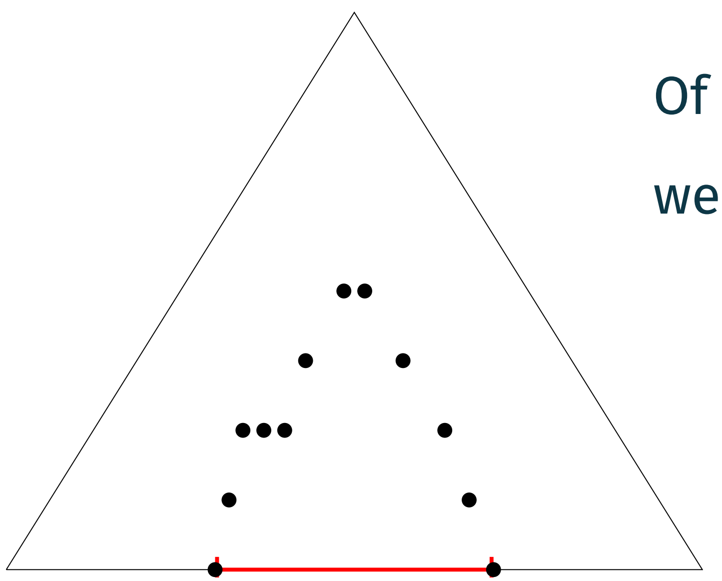

Of course, periodicity causes trouble:

we have $\mathcal{O}(\log N)$ groups of breakpoints.

PILLAR Operations in the Compressed Setting

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After a preprocessing that runs in time nearly-linear in the size of the SLPs that generate P and T, we can perform each PILLAR operation in $\mathcal{O}(\log^2 N \log\log N)$ time.

[I, CPM'17; CKW'20]

Approximate Pattern Matching under Edit Distance: Summary of Results

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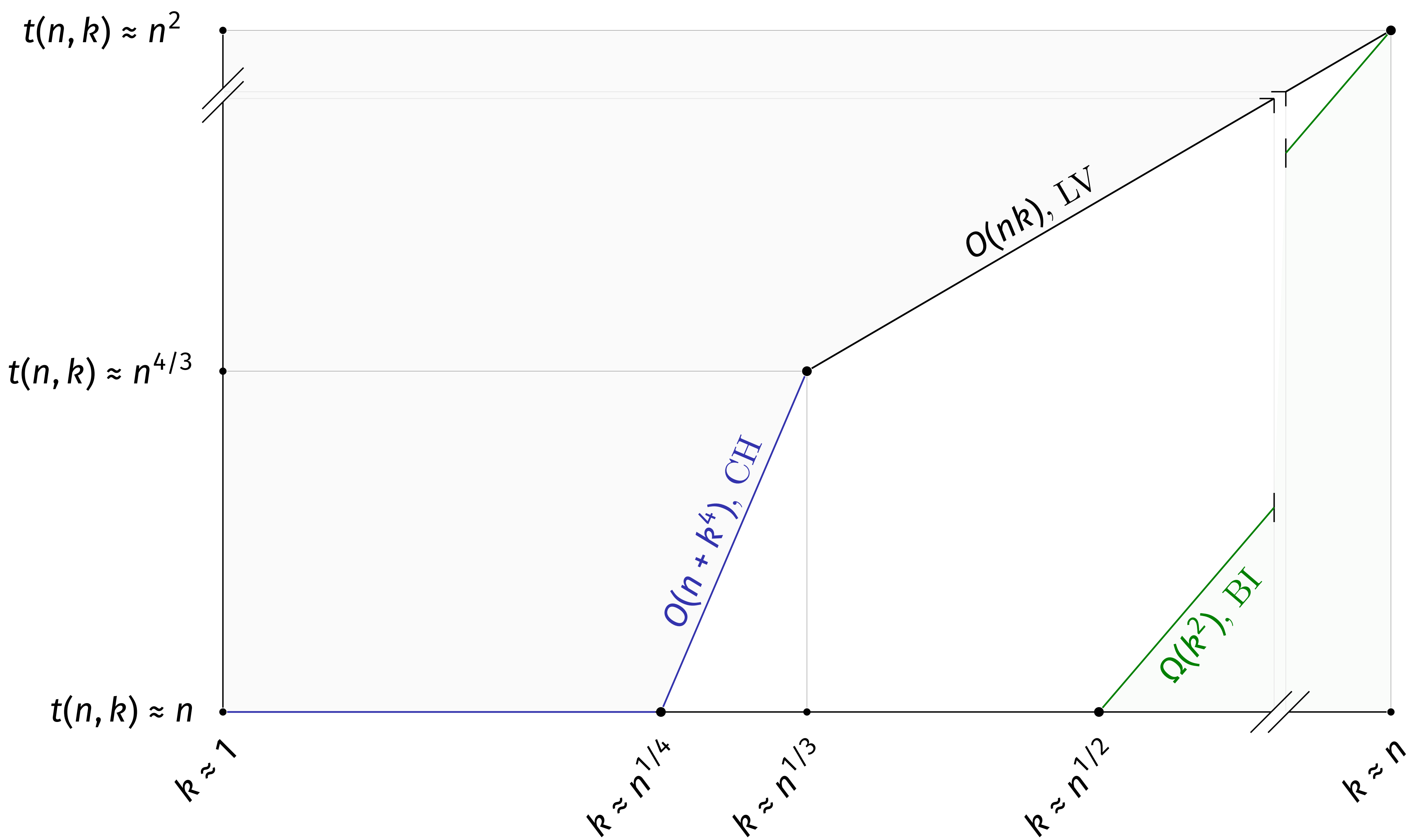

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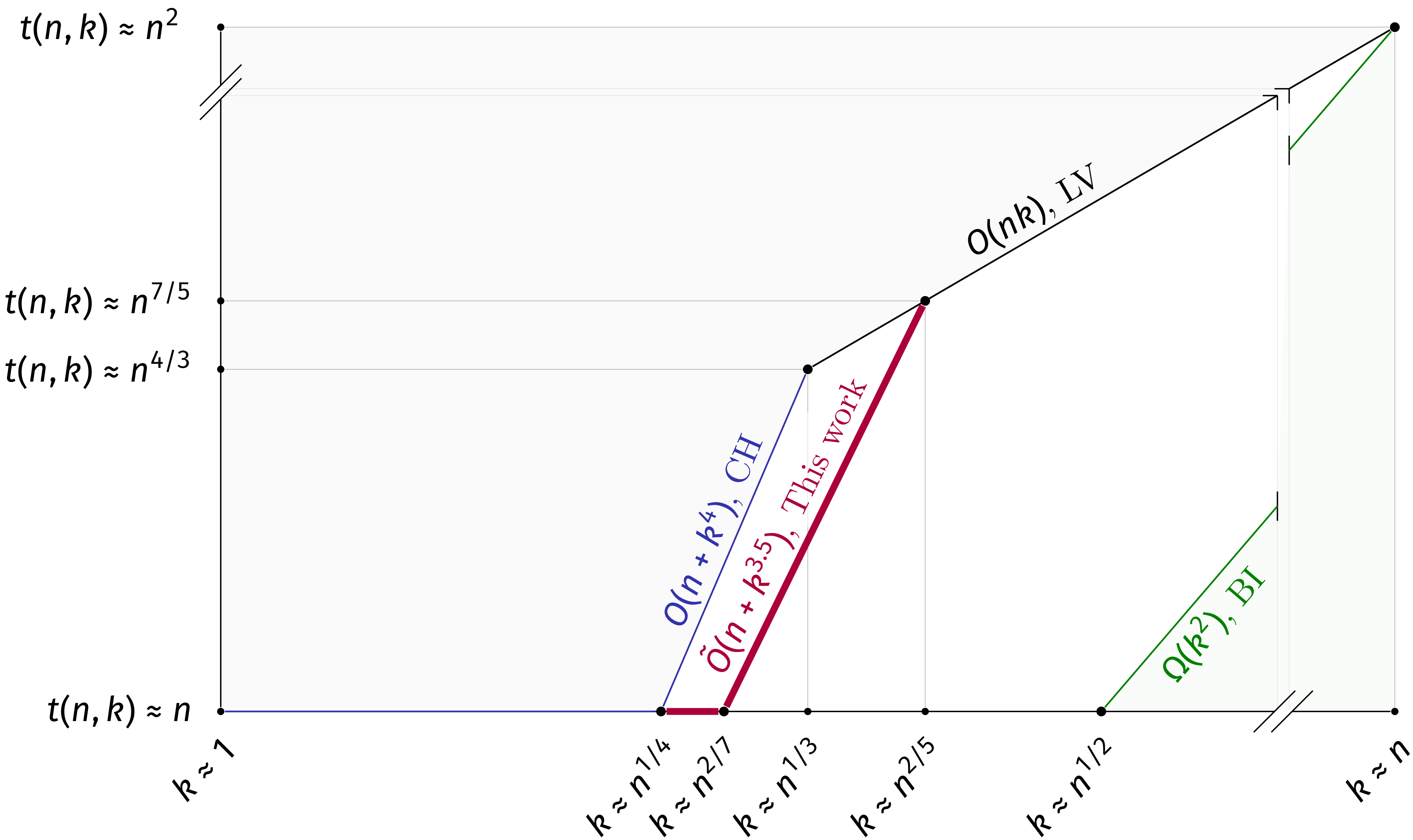

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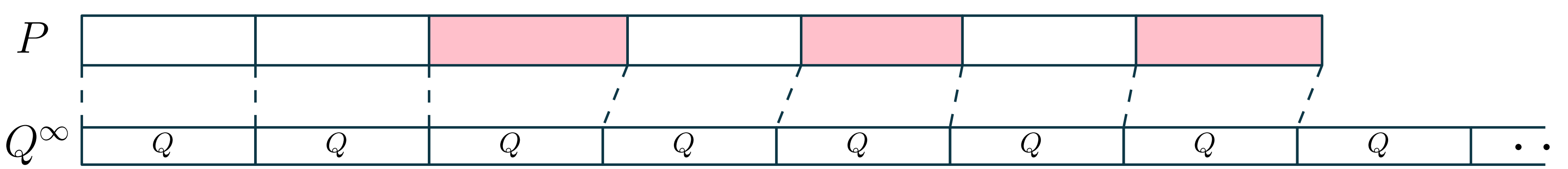

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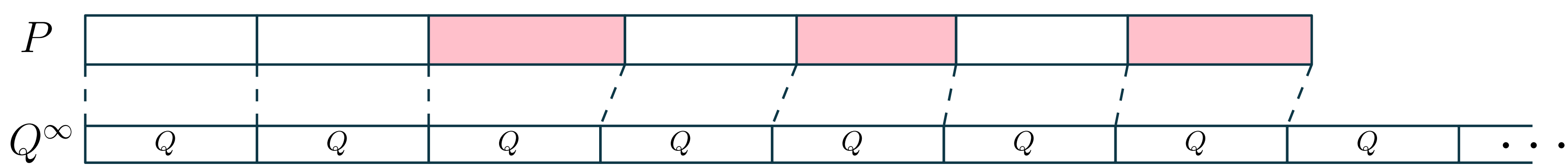

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We call this a tile decomposition of P with respect to Q.

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Reduction [CKW'20]: An algorithm that solves the almost periodic case in $\tilde{\mathcal{O}}(k^a \cdot n/m)$ PILLAR-time, for $a \geq 3$, implies an algorithm that solves the general case in $\tilde{\mathcal{O}}(k^a \cdot n/m)$ PILLAR-time.

Dynamic Puzzle Matching

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After $\tilde{\mathcal{O}}(k^3)$-time preprocessing, updates and queries take $\tilde{\mathcal{O}}(k)$ time.

Using Dynamic Puzzle Matching

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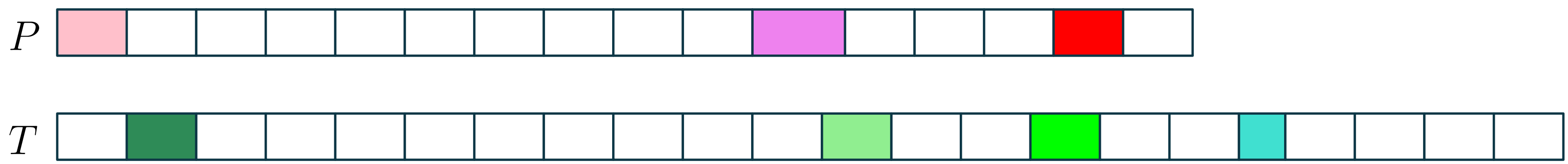
Think of: $k = 4$ and $|Q| \approx \sqrt{m}$.



Each string has $\mathcal{O}(k)$ special tiles.

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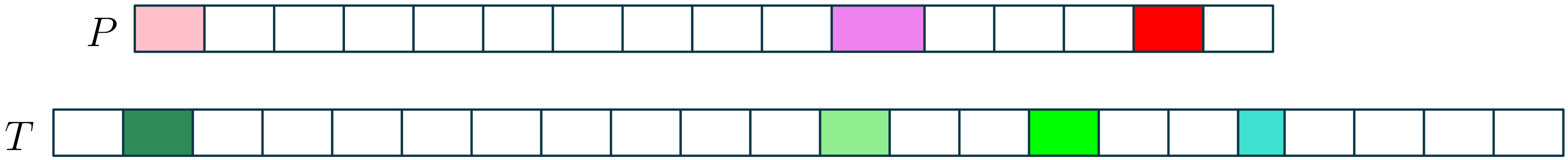

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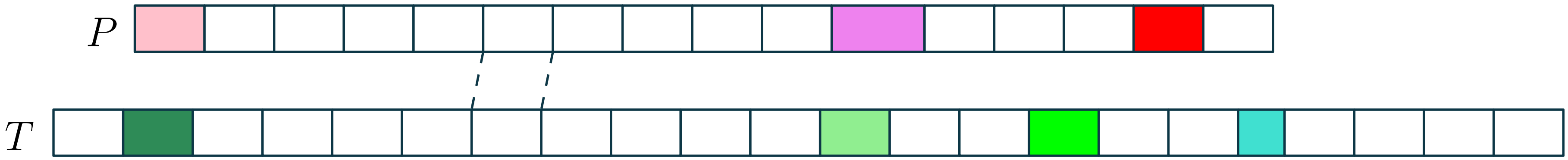

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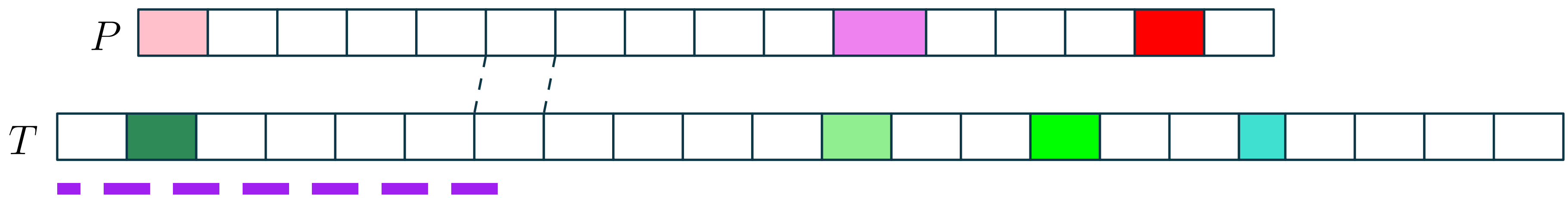

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Starting positions of k-error occs in T are within $\mathcal{O}(k)$ from endpoints of tiles.

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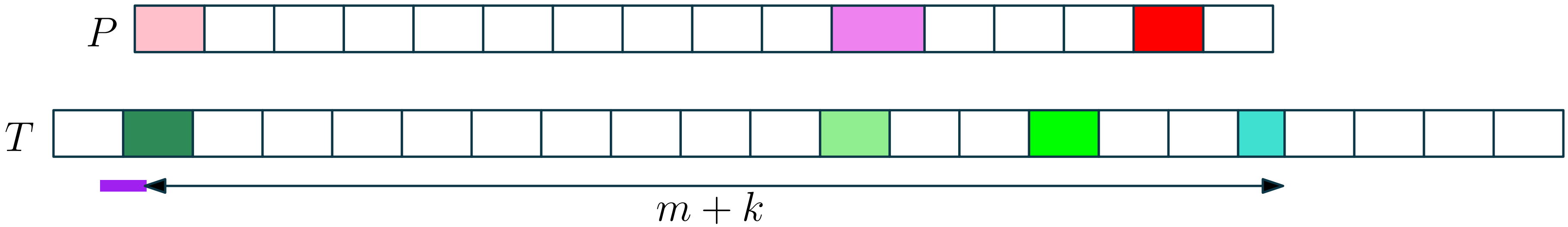

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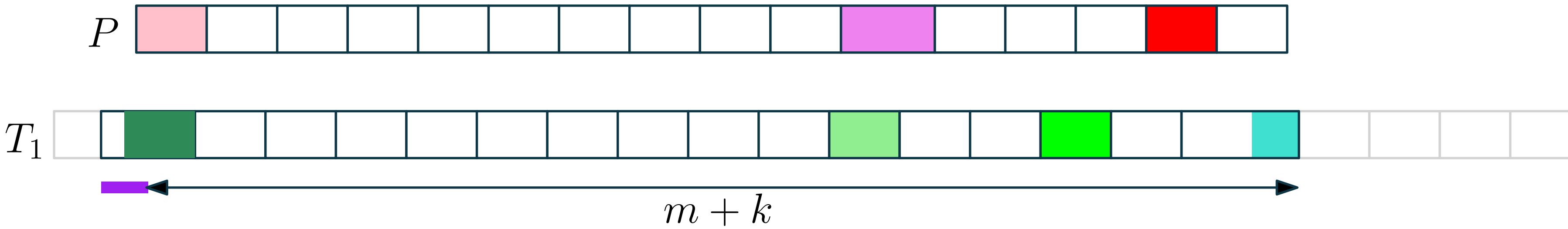

$$|T_j| = m + \mathcal{O}(k)$$

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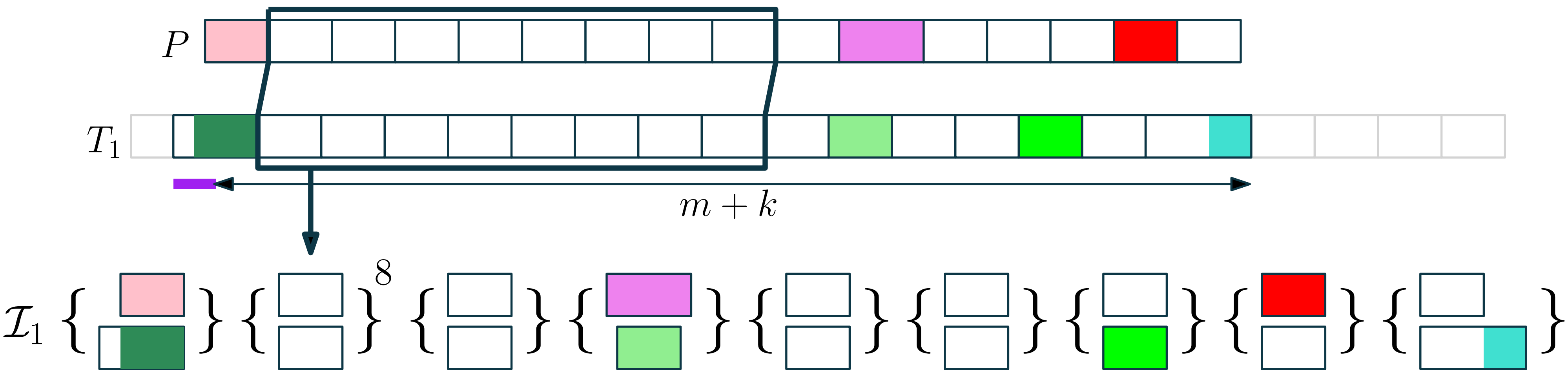

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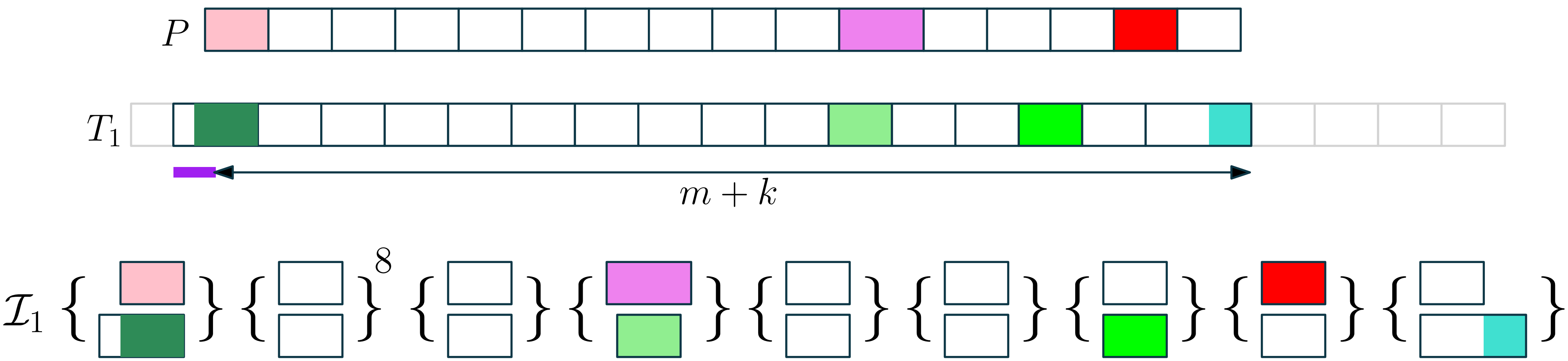

Goal: Iterate over all $\mathcal{I}_j$'s in a DPM instance.

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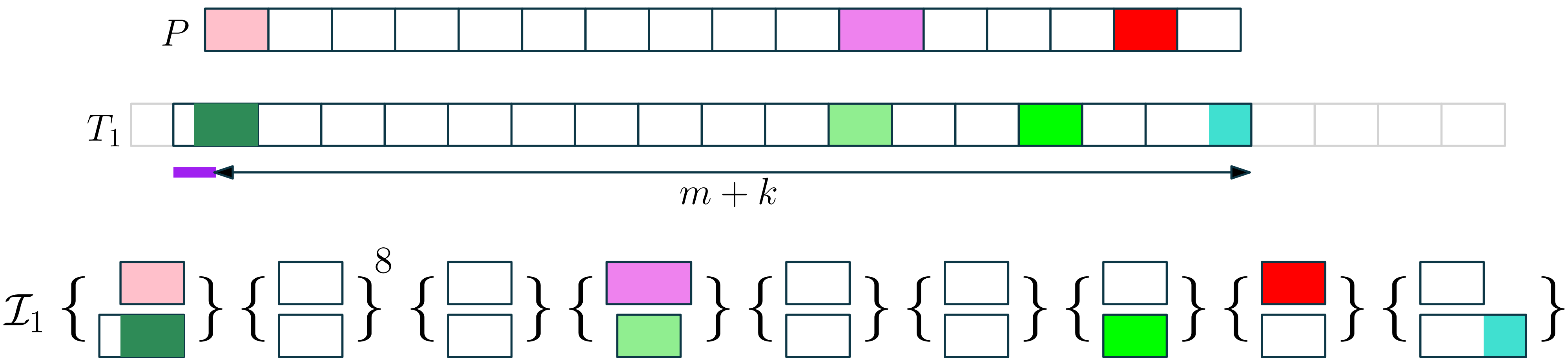

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(The leading and trailing pairs are treated separately.)

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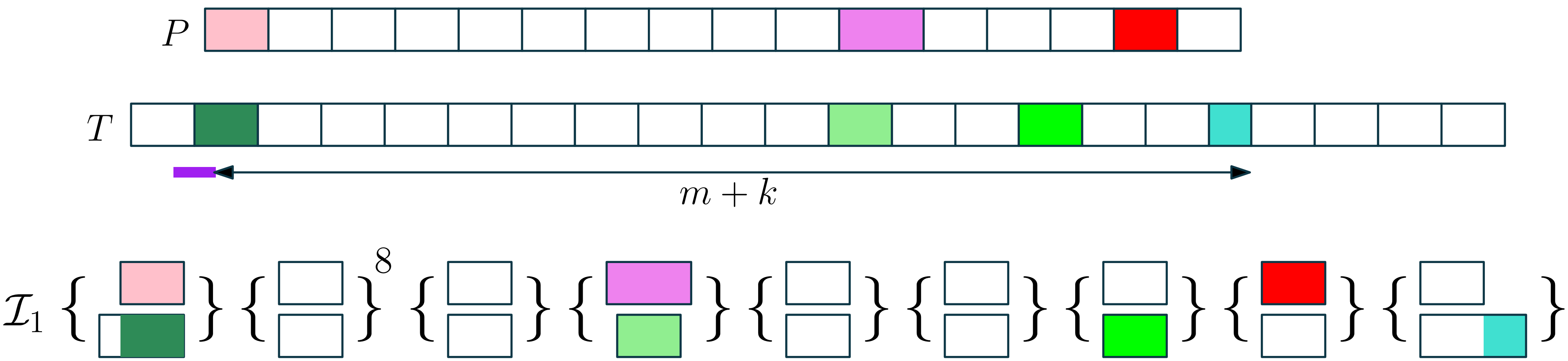

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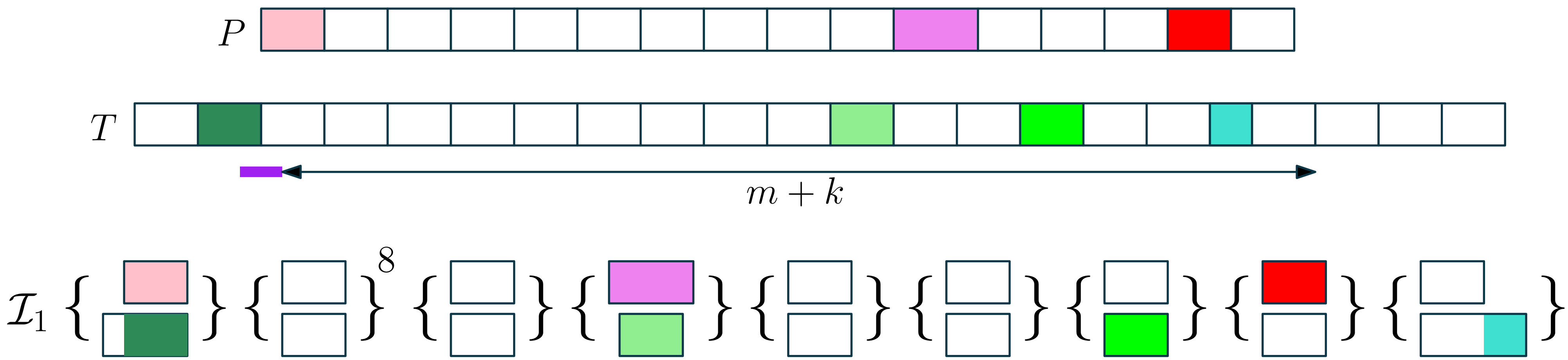

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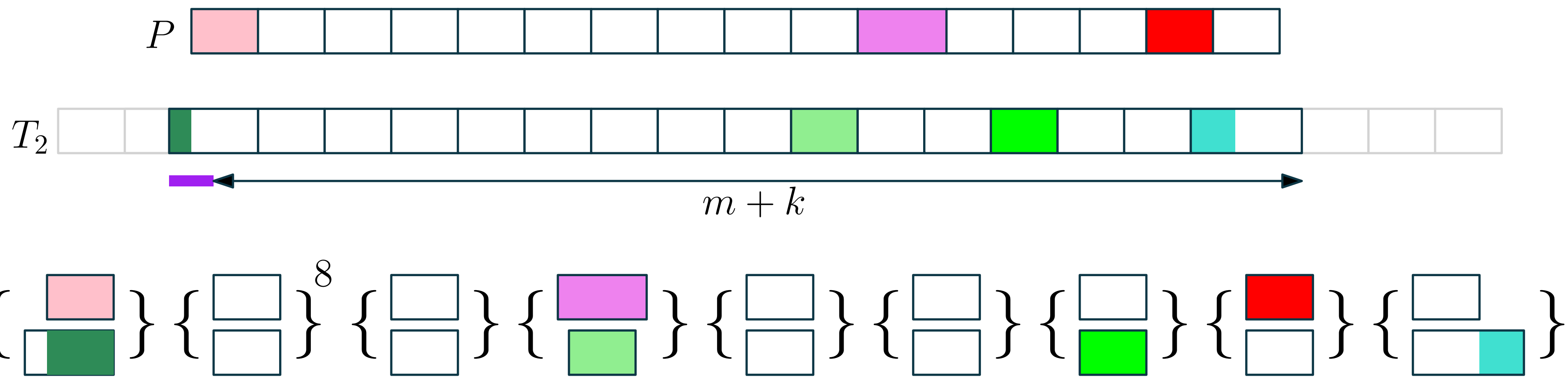

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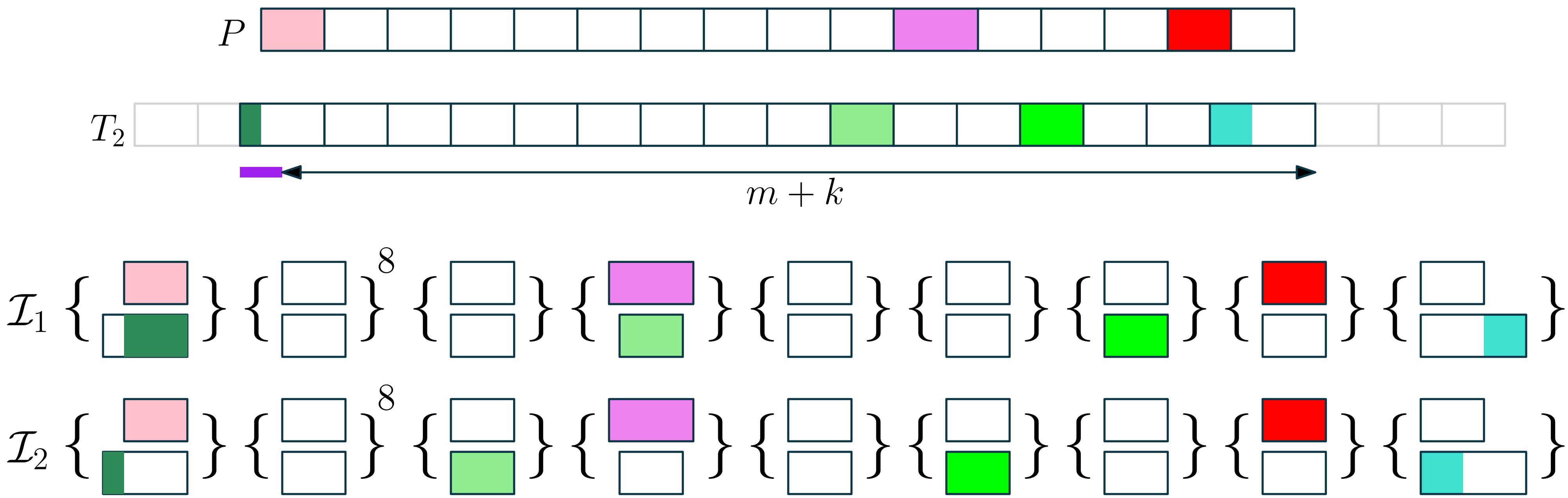

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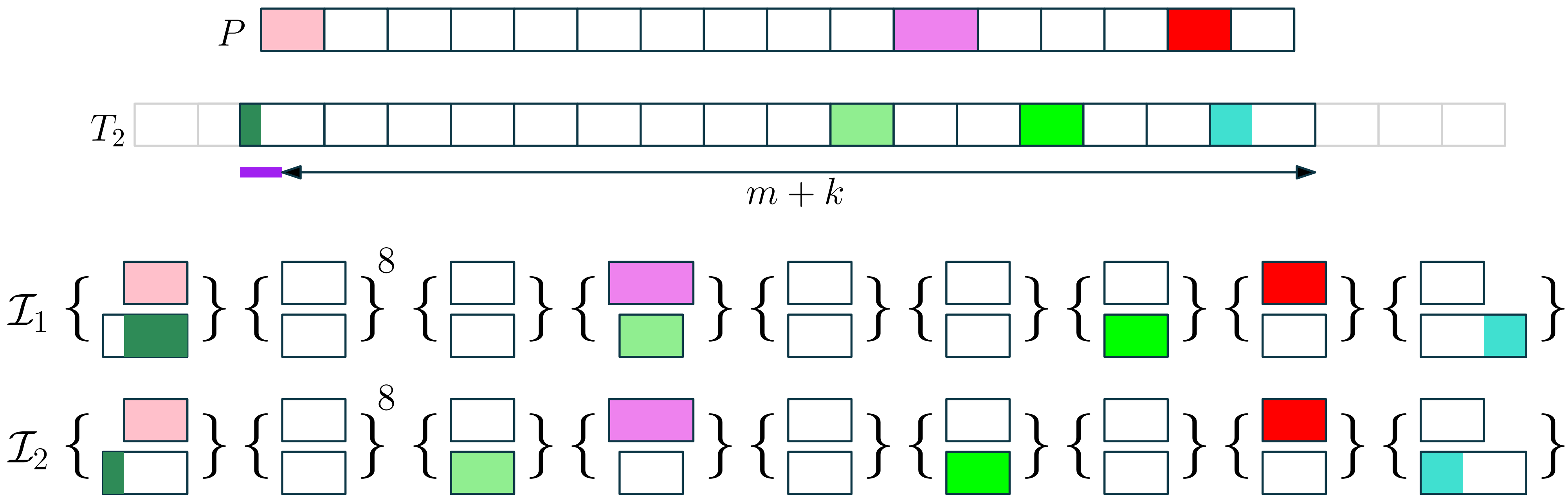

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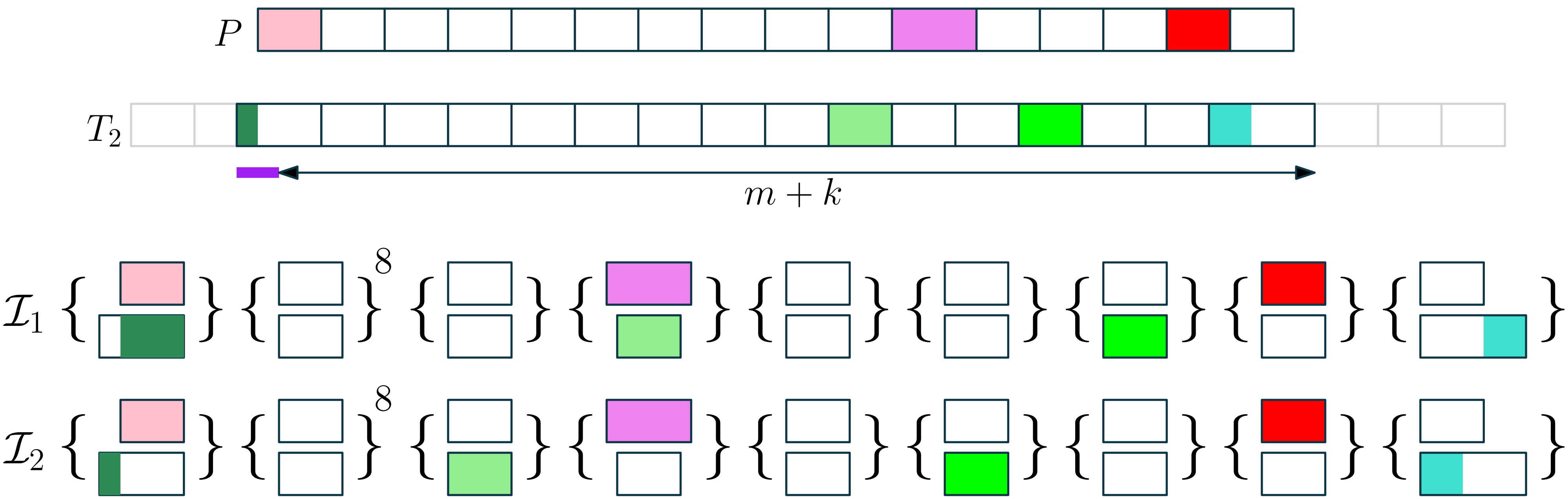

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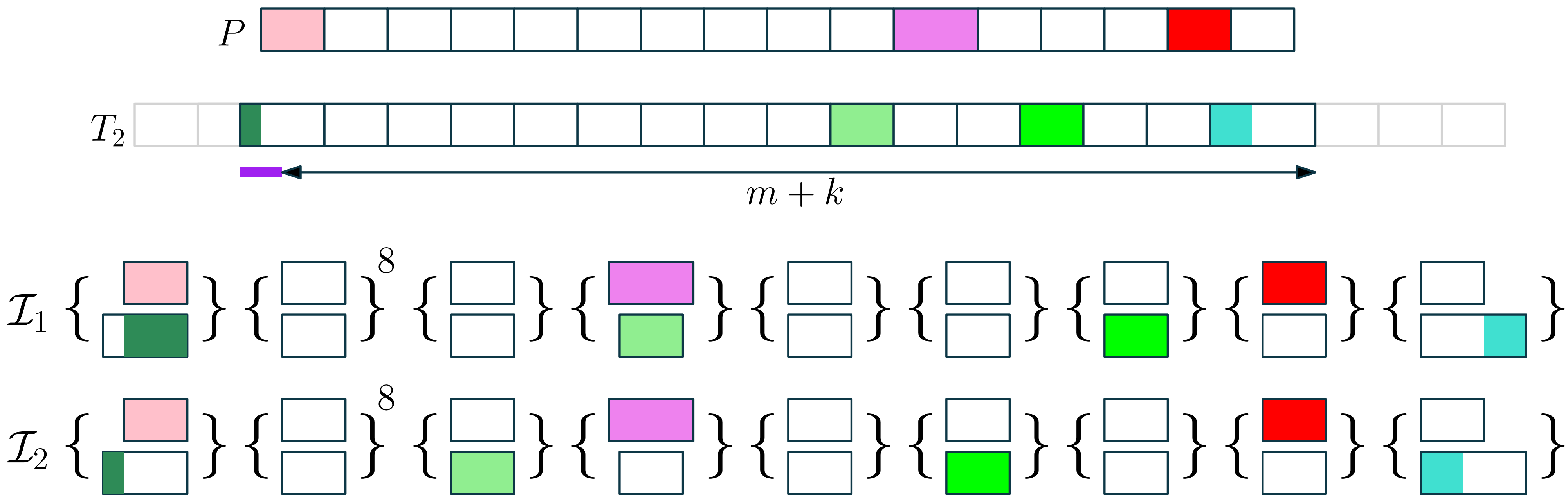

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Yields $\tilde{\mathcal{O}}(k^3 + \sqrt{m} \cdot k^2)$.

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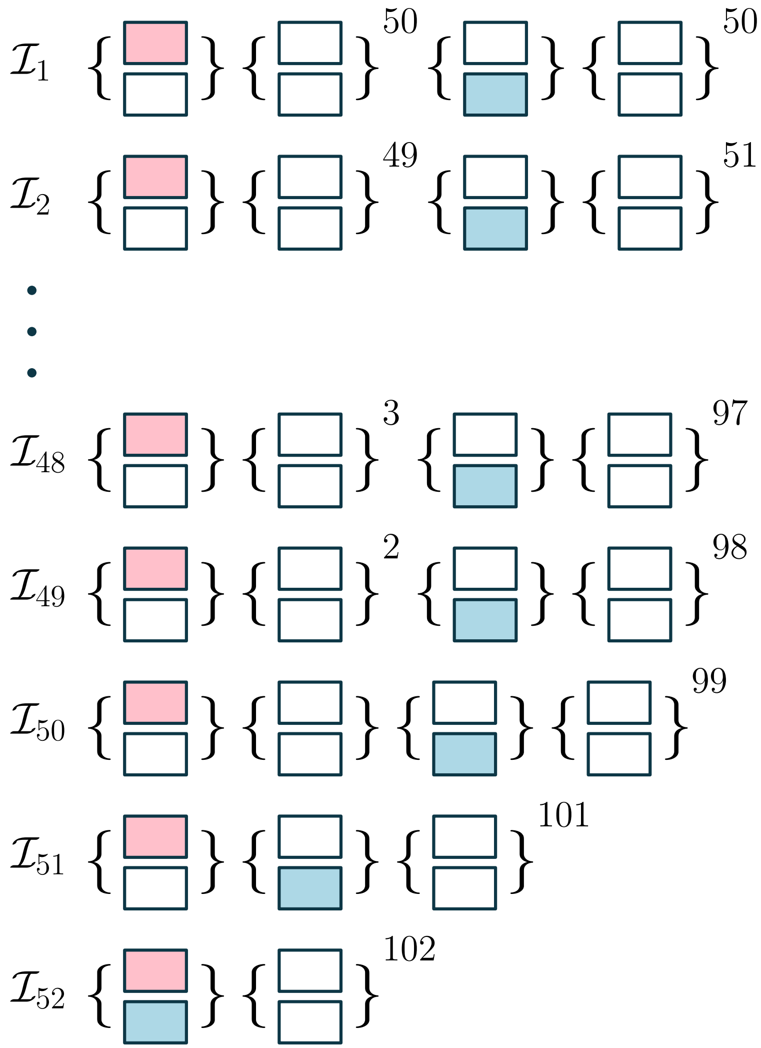

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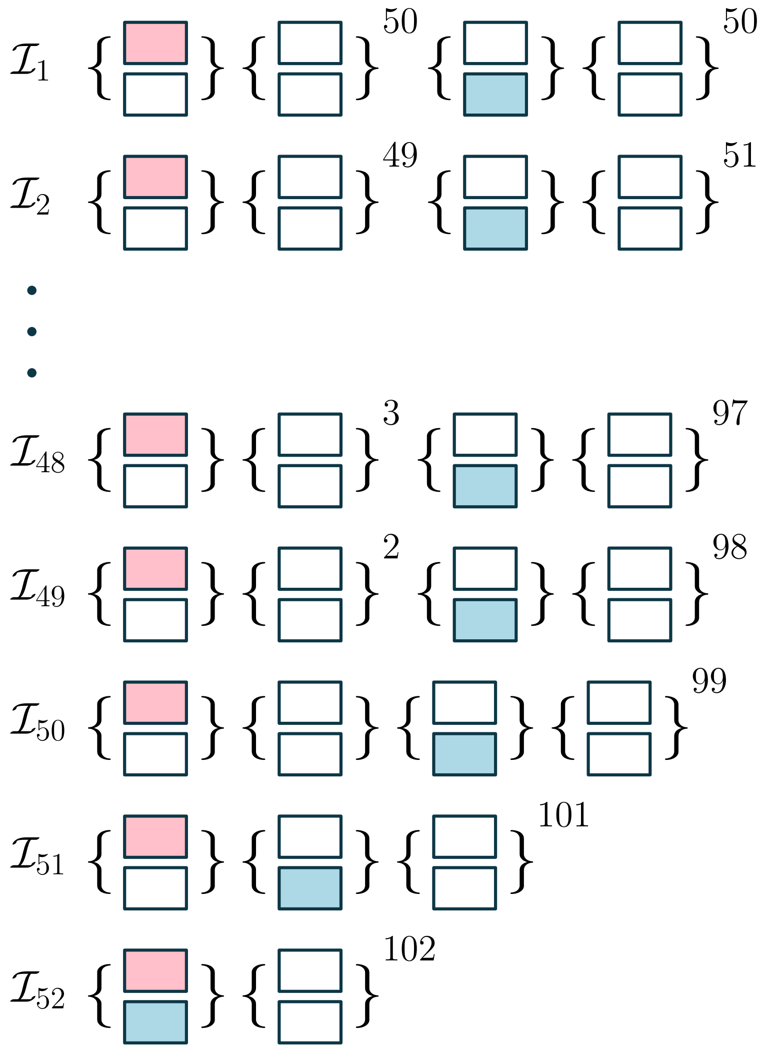

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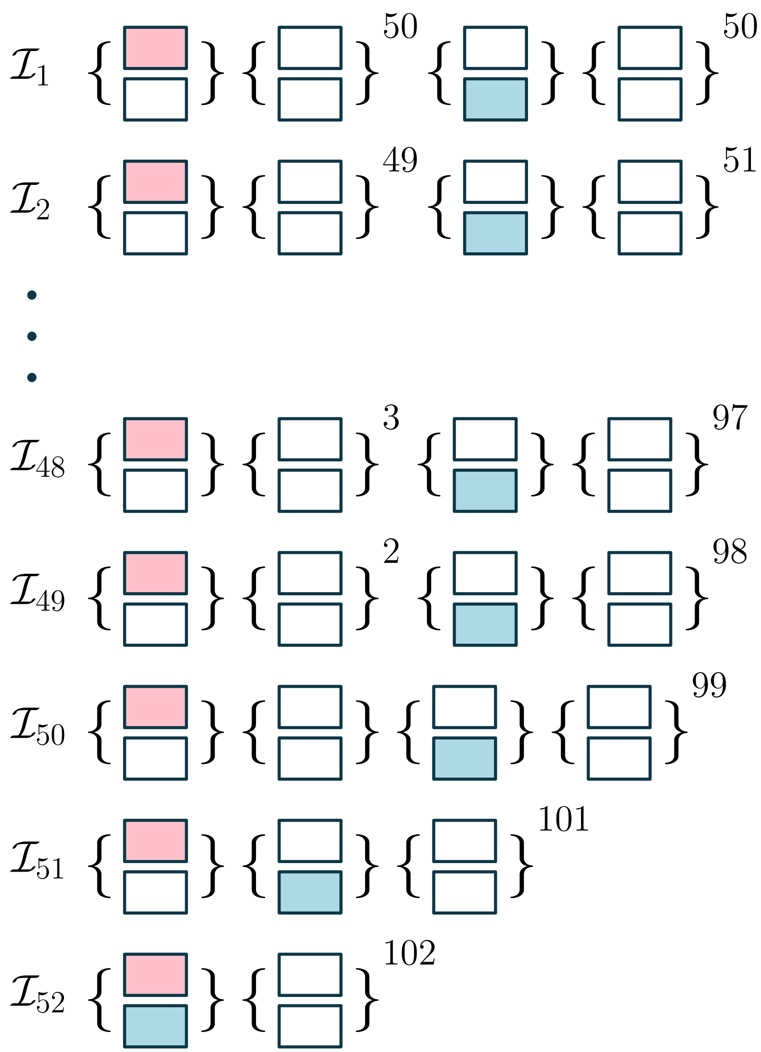

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Cap exponents of plain runs at $k + 1$.

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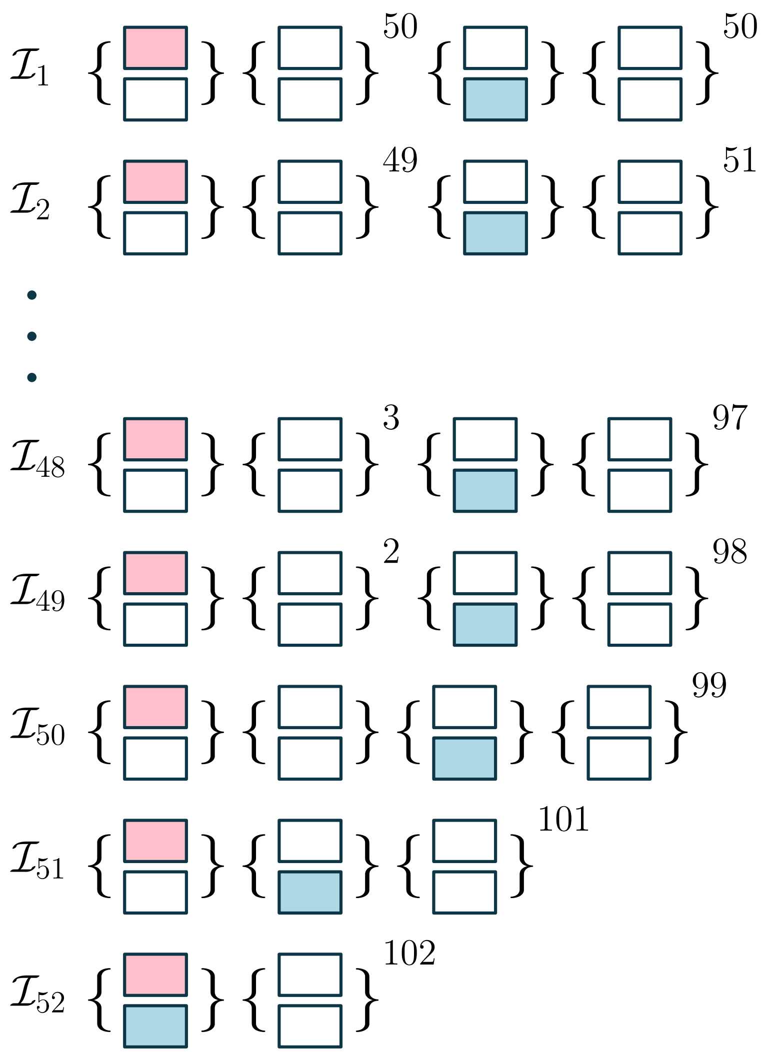

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We do not lose or gain any k-error occs.

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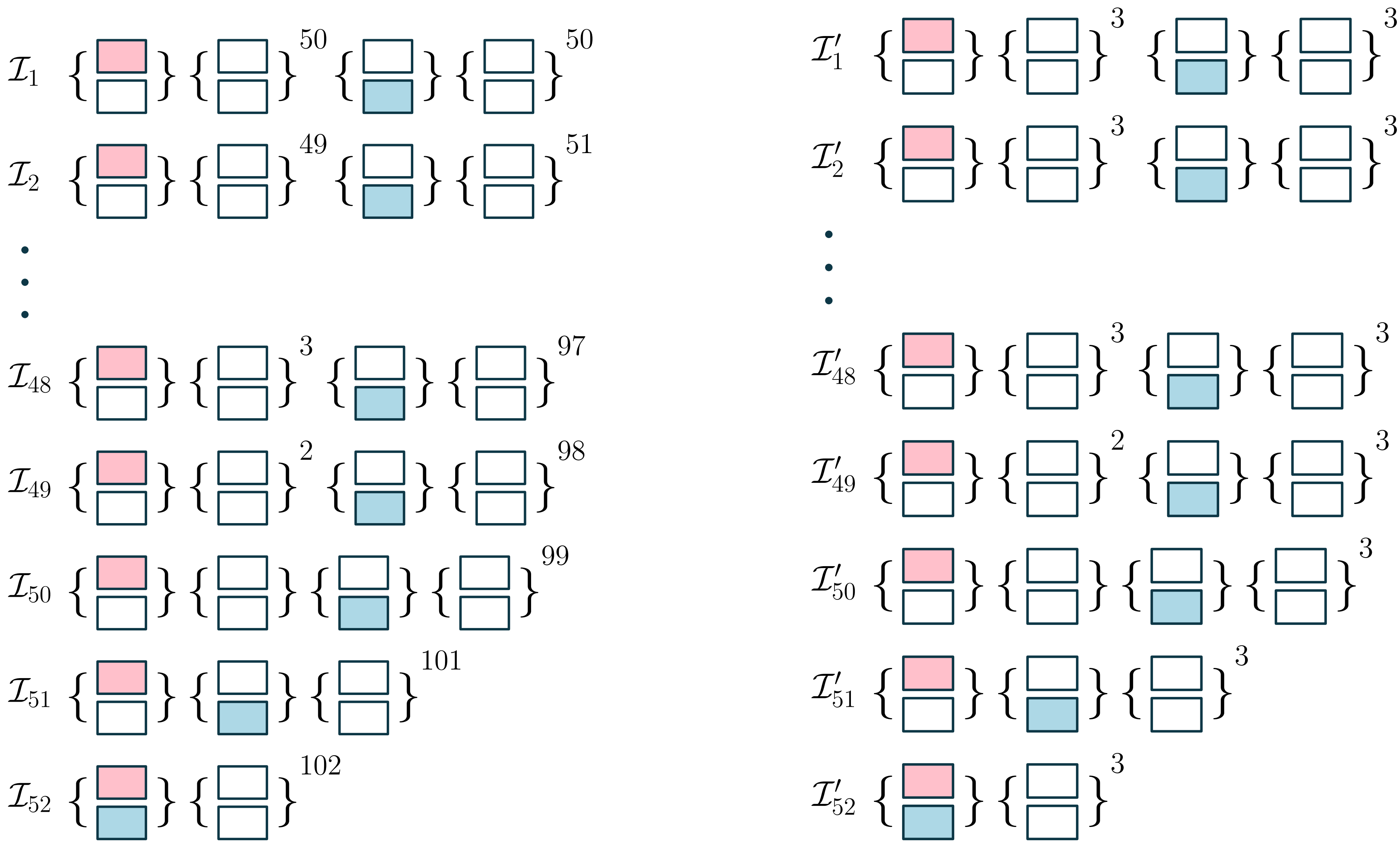

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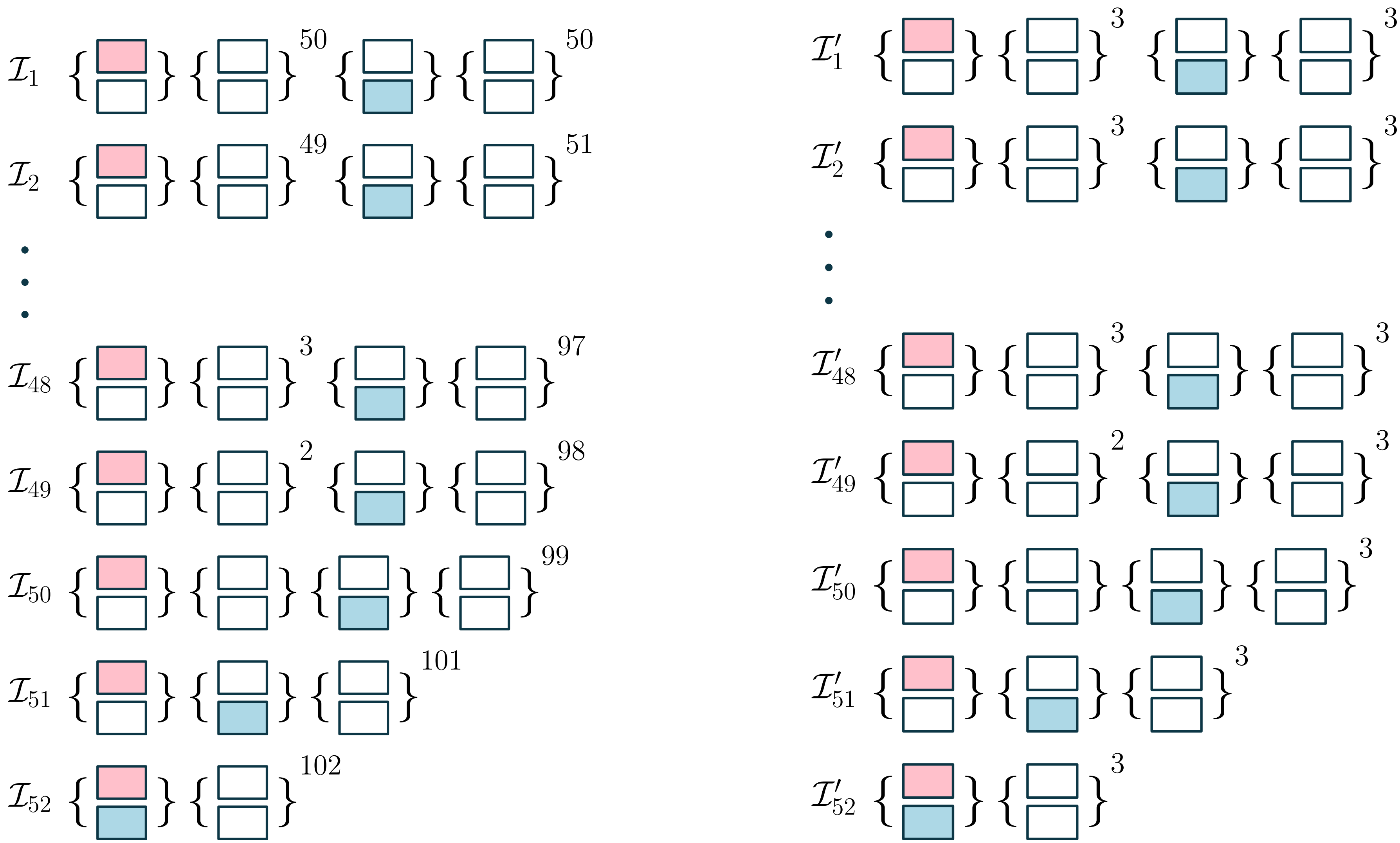

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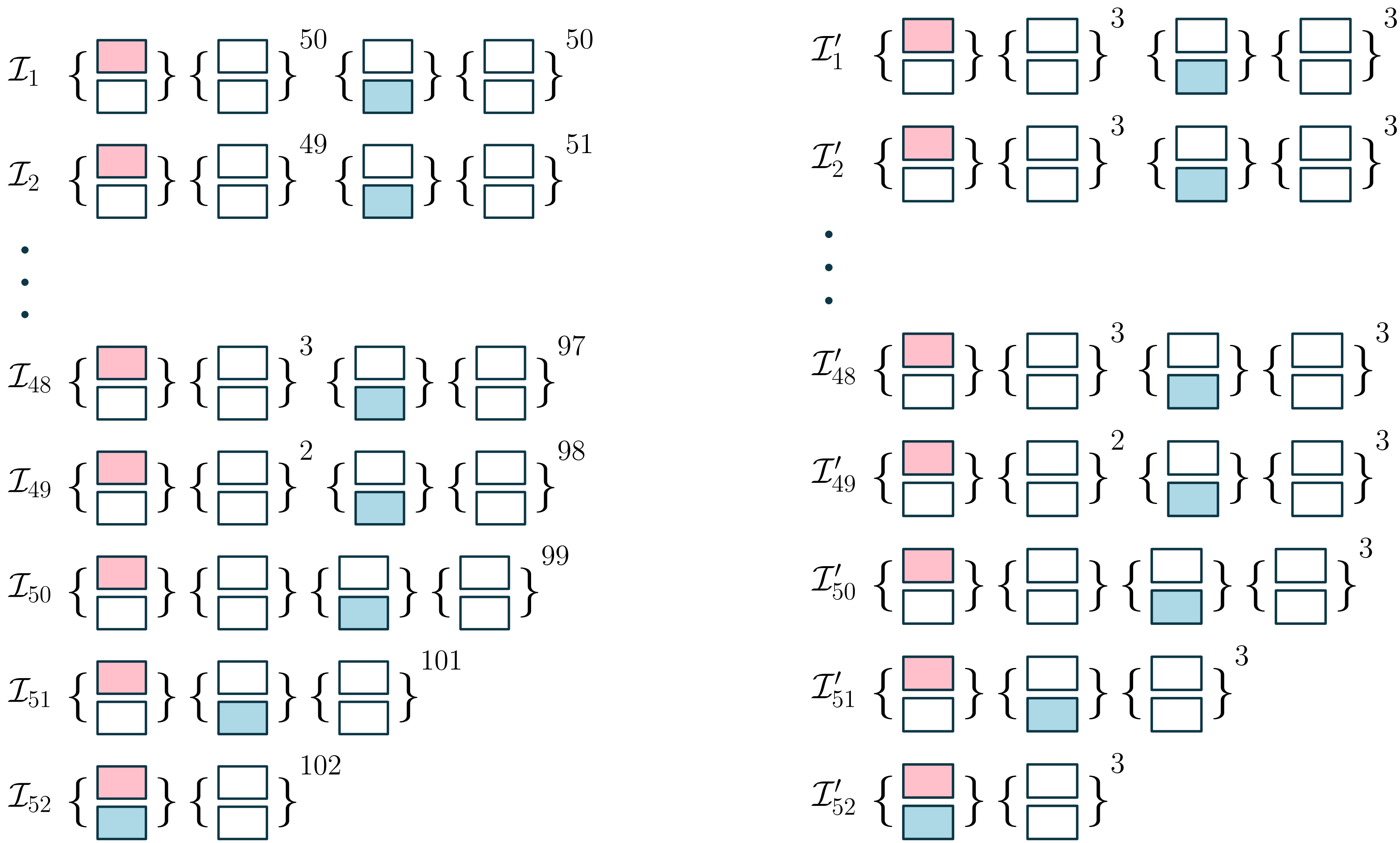

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We have $\mathcal{O}(k^2)$ pairs of special tiles!

$\mathcal{O}(k^3)$ DPM-updates via Primitivity

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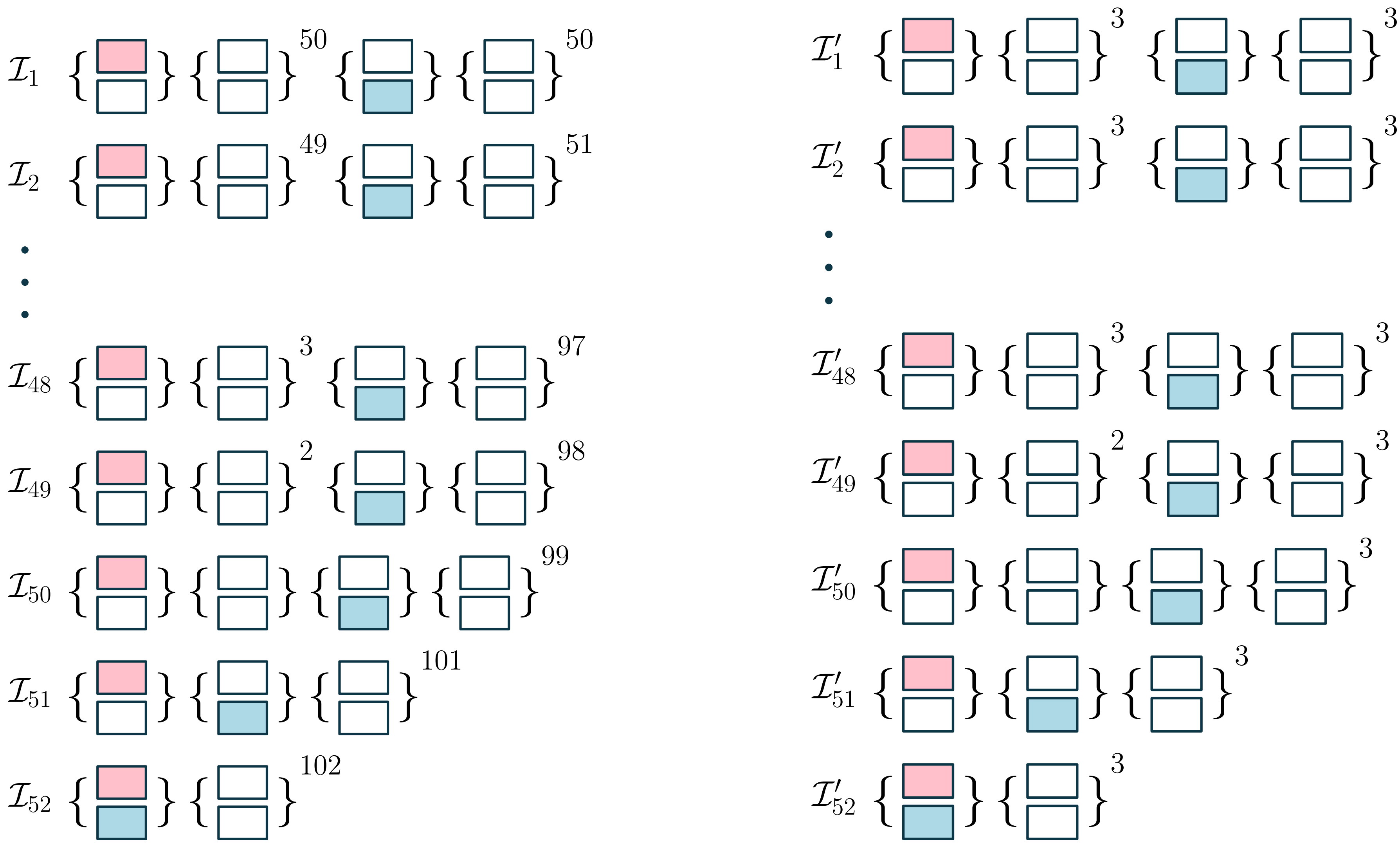

Alternative $\tilde{\mathcal{O}}(k^4)$-time algorithm!

Overview for $\mathcal{O}(k^{2.5})$ DPM-updates

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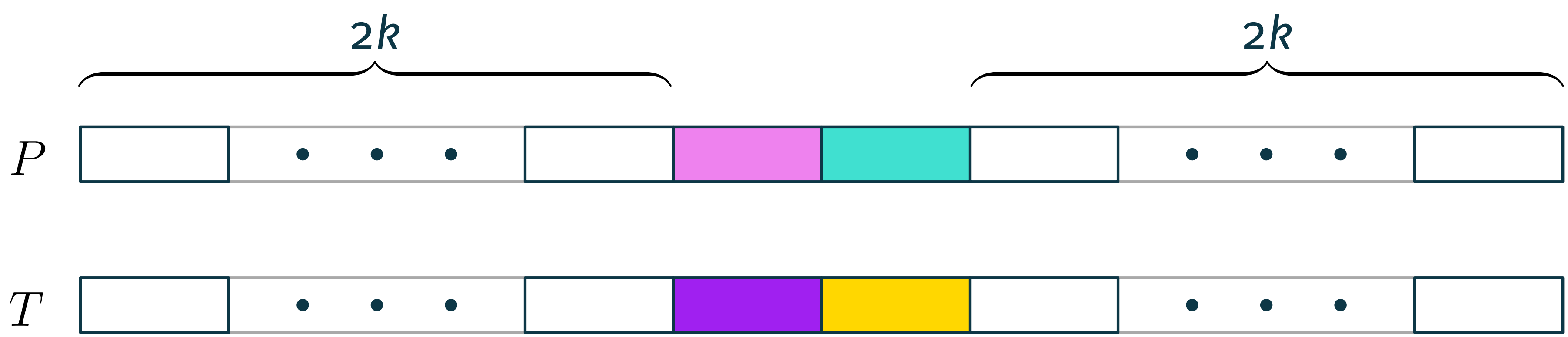

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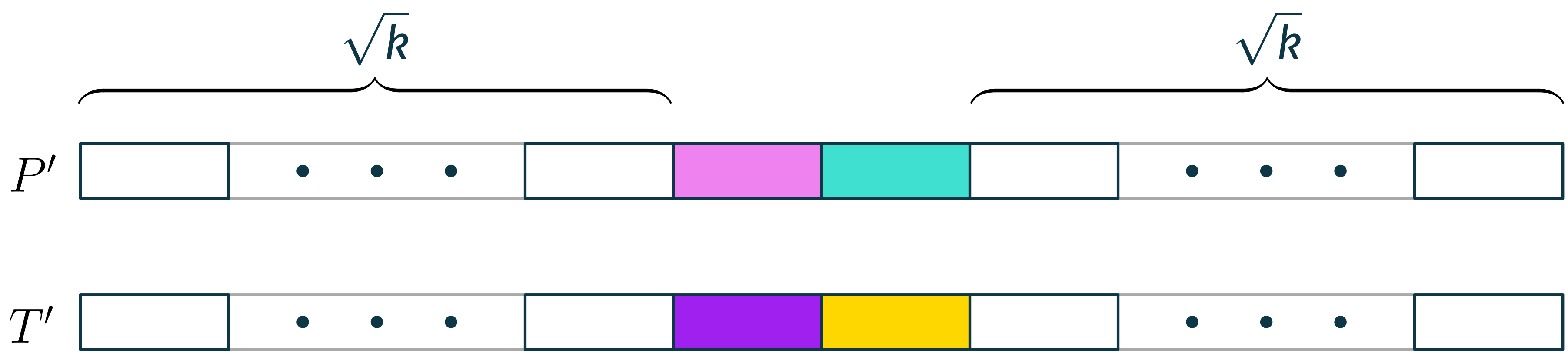

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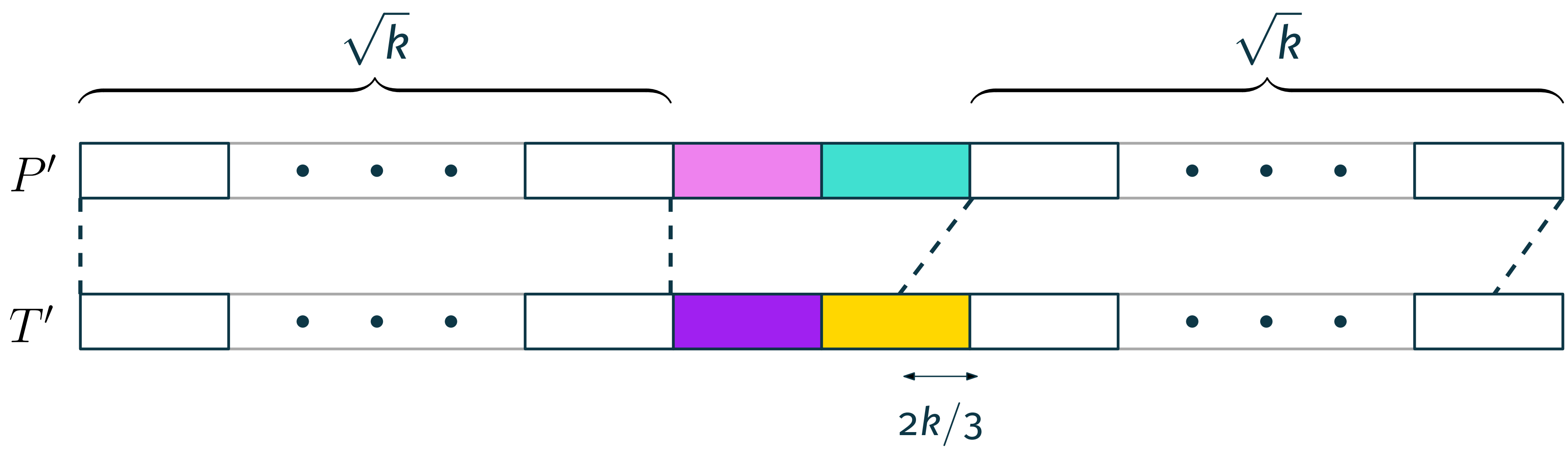

Cost: $0 + (k/3 + 1) + \sqrt{k} \cdot \delta_E(Q, rot^{2k/3}(Q))$.

Overview for $\mathcal{O}(k^{2.5})$ DPM-updates

Cap exponents of plain runs at $\sqrt{k}$.

We may get false positives when we have $\geq \sqrt{k}$ edits in a run of (Q, Q).

In this case, we must be saving $\geq \sqrt{k}$ by canceling out errors between P and Q^∞ with errors between T and Q^∞.

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This yields $\mathcal{O}(k^{2.5})$ DPM-updates and hence $\tilde{\mathcal{O}}(k^{3.5})$ time overall.

A Solution to DPM and a Grid View

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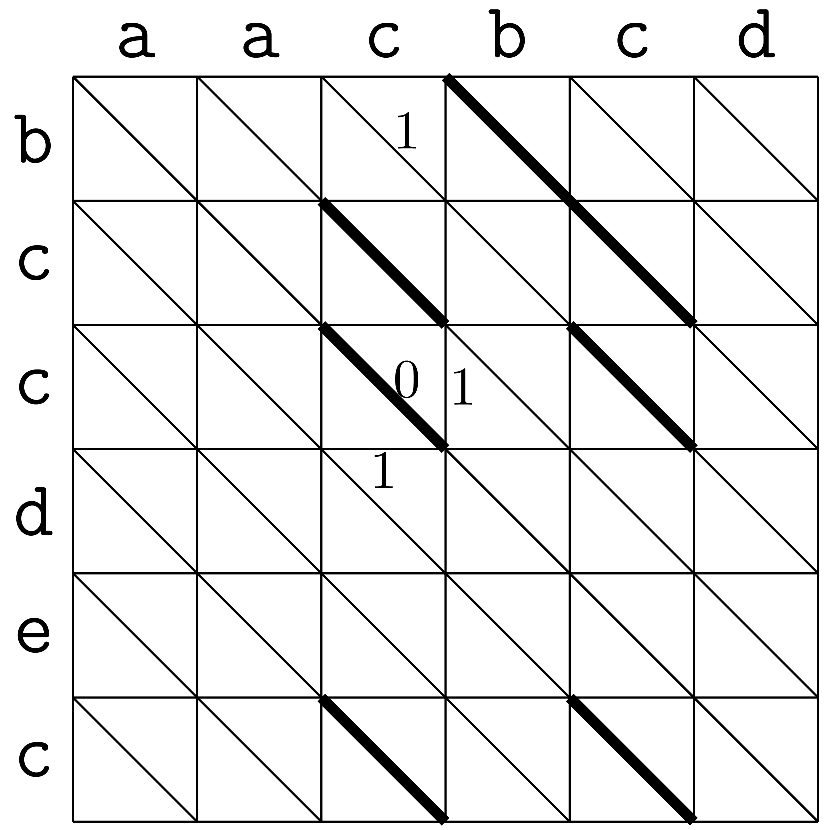

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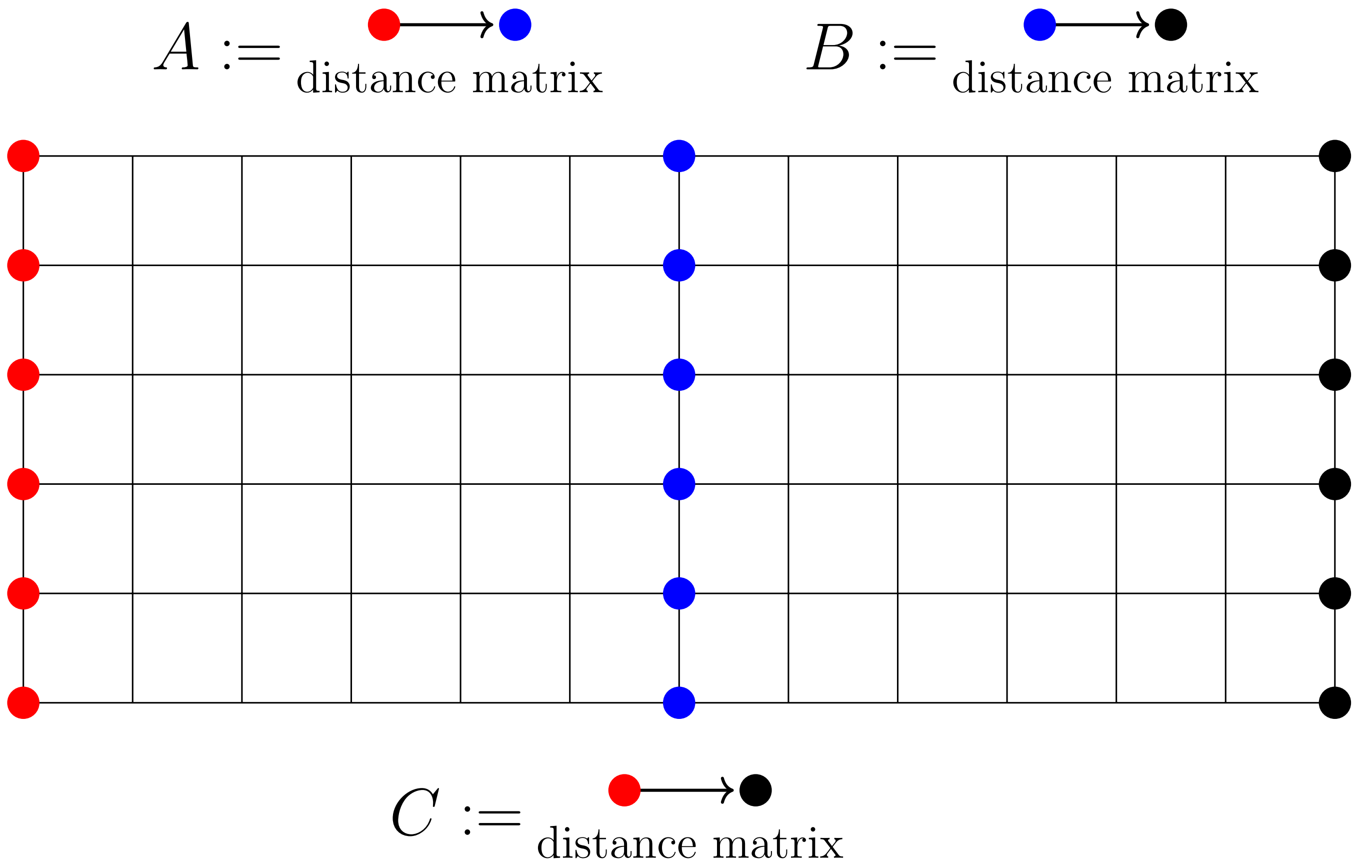
Theorem [Tiskin; Algorithmica 2015] Matrix C can be computed from (small representations of) $n \times n$ matrices A and B in $\mathcal{O}(n \log n)$ time.



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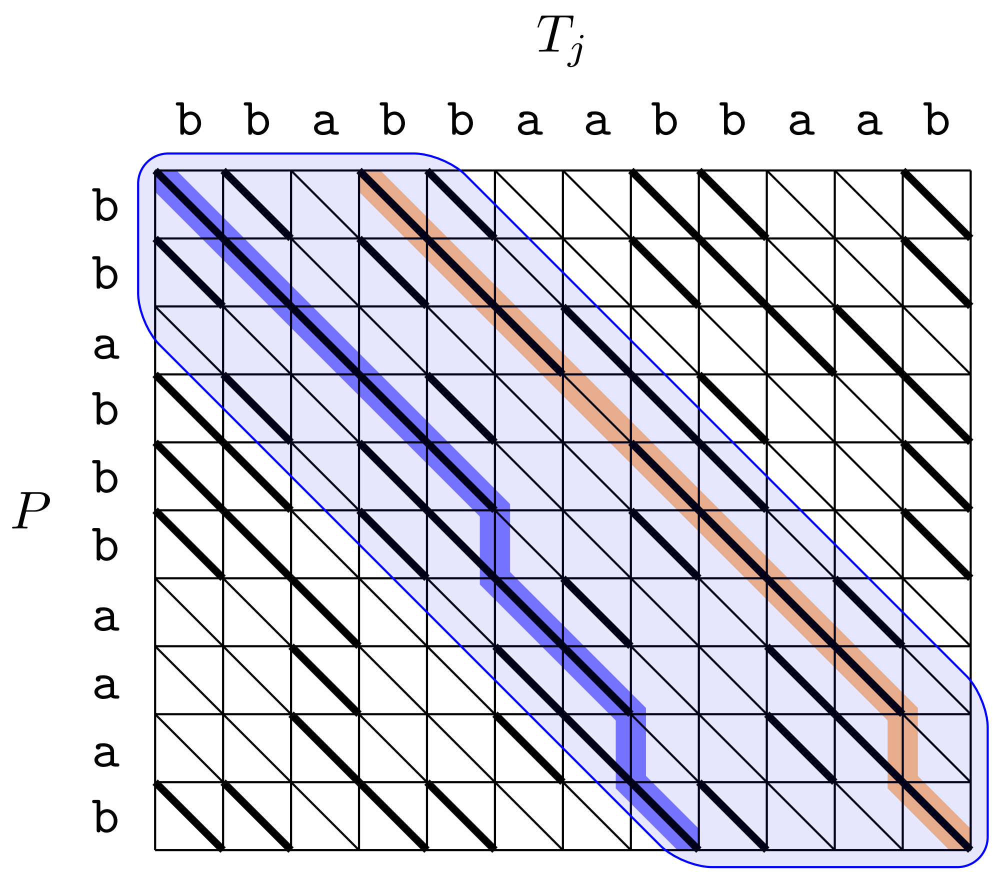

$$P = 10,\ T_j = 12,\ k = 2.$$

Only $|T_j| - |P| + 2k + 1 = \mathcal{O}(k)$ diagonals are relevant.

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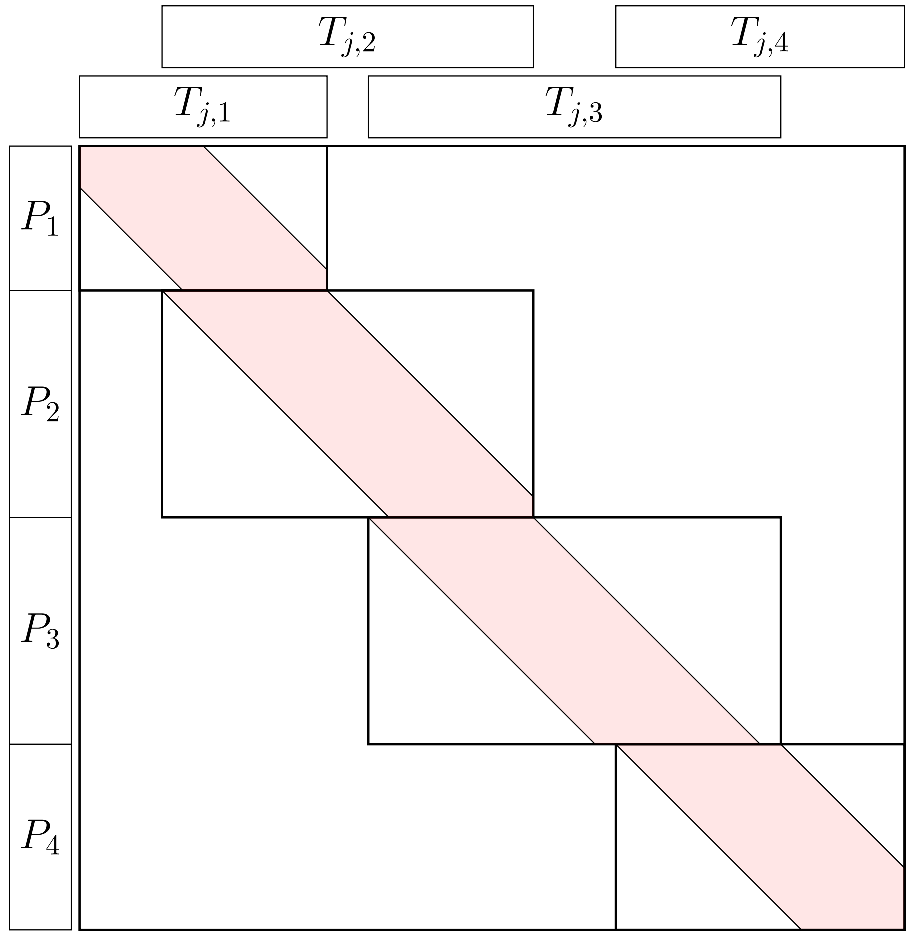

Preprocessing: Build distance matrices for these small alignment grids.

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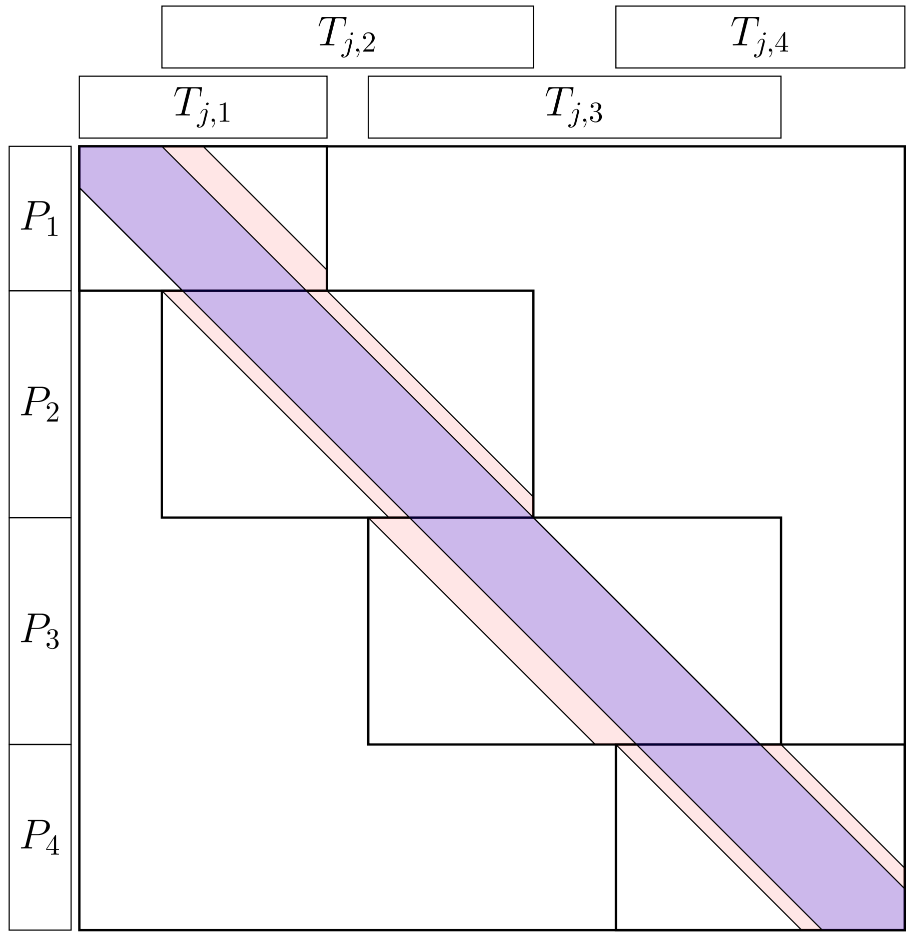

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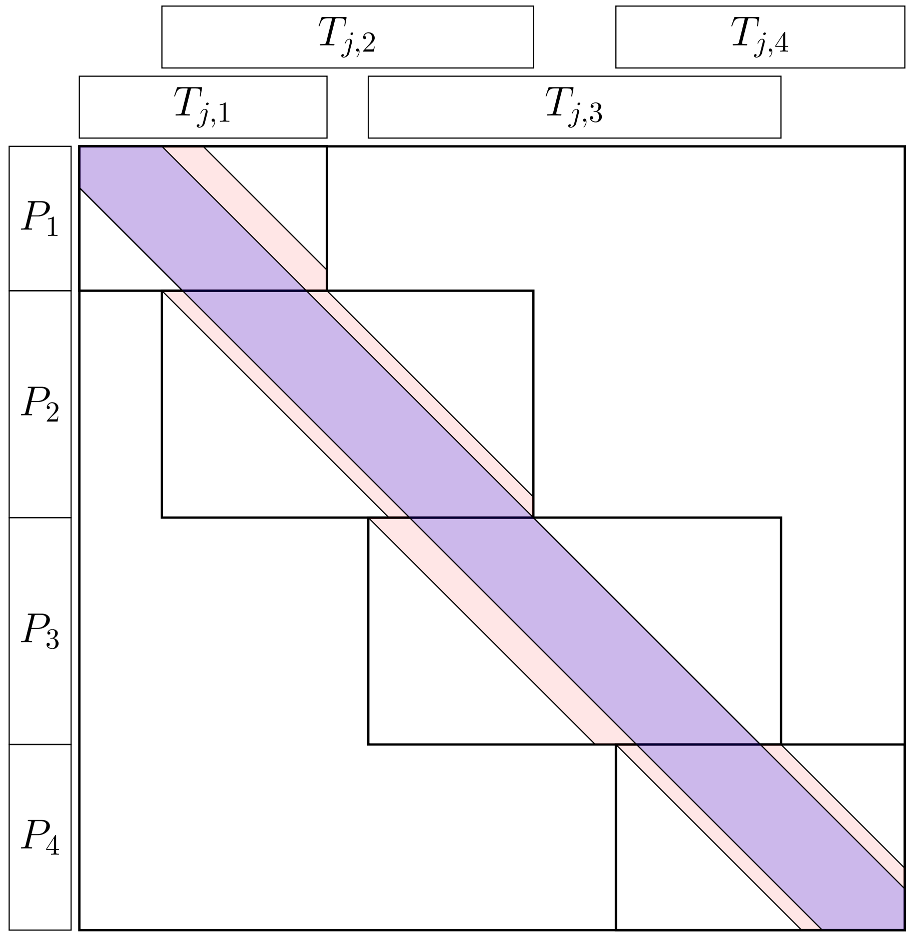

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Each stitching operation takes $\tilde{O}(k)$ time.

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We report starting positions. How fast can we report substrings?

The End

Thank you for your attention!

Many thanks to Philip Wellnitz for sharing his slides from SODA 2019!

I have edited the portion I used, so I am responsible for any errors. :)